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The Using of Analogy Between Lorentz's Calibration for Electromagnetic Fields and de Donder-Fock Condition for Determining Global Characteristics of Gravitational Fields

Contents

Abstract

The global dynamic characteristics for Schwarzschild and Kerr solutions of Einstein's equations are obtained by using harmonic coordinates as an analogy of Lorentz's calibration in electromagnetic theory. For these asymptotically flat systems the Papapetrou's pseudotensor turns out to be singularly useful due to its bimetrical nature in the background of a flat space-time. The obtained results confirm the physical interpretation of these metrics. The method developed serves as an element for determining the physical nature of the sources generating these gravitational fields. It can also be applied, under certain conditions, for the determination of the global dynamic characteristics for some axial symmetrical insular stationary exact solutions of Einstein's equations.

1. Introduction

The physical meaning of the solutions of the Einstein's equations is conditioned partially by determination of the global dynamic characteristics of these fields, which at the same time can give us clues upon the physical nature of the sources originating the gravitational fields described by these solutions. However the definition of the density of these quantities that is the energy-momentum density of the matter-gravitation system is a problem that has not been understood completely due to the geometrical approach of Einstein's general relativity theory (GRT). It seems natural that in any concrete analysis in the framework of the GRT it should always be necessary to fix the coordinate system. This is a consequence from the fact that Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}, \tag{1}$$

are equations for 10 independent functions of the field $g_{\mu\nu}$, however, due to the four Bianchi identities

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{;\nu} = 0 \tag{2}$$

it so happens that there are only six equations for ten unknown metric functions. This means that in the

10 components $g_{\mu\nu}$ there are four degrees of liberty. These correspond exactly to the arbitrariness of the coordinate system in which the tensor metric is given, that is, to the arbitrary transformation of coordinates

$$g'^{\mu\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x). \tag{3}$$

This transformation introduces four independent functions corresponding just to the four degrees of liberty in the solution of Einstein's equations.

Note that a direct analogy with the definition in electrodynamics of the vectorial potential exists. In fact, the Maxwell's equations for the vectorial potential have the form

$$\Box A_{\alpha} - A^{\beta}_{,\alpha\beta} = -j_{\alpha}. \tag{4}$$

However, besides these four equations for the four unknown functions A_{α}, it is necessary to consider the charge conservation law

$$j^{\alpha}_{;\alpha} = 0. \tag{5}$$

From this it follows that

$$\left[\left(\Box A_{\gamma} - A^{\beta}_{,\beta\gamma}\right) g^{\alpha\gamma}\right]_{,\alpha} = 0 \tag{6}$$

and therefore it has a one degree of liberty. This last one corresponds to the gradient transformation

$$A'_{\alpha} = A_{\alpha} + \frac{\partial \Phi}{\partial x^{\alpha}}. \tag{7}$$

This indetermination is eliminated by concrete calibration choice, for example by Lorentz's calibration

$$A'^{\alpha}_{,\alpha} = 0. \tag{8}$$

with which the scalar Φ is fixed

$$\Box \Phi = -A^{\alpha}_{,\alpha}. \tag{9}$$

If we carry this analogy further, one can propose a condition upon the coordinates as an expression like (8), taking as potential the density of the contravariant metric tensor (De Donder-Fock's condition)

$$\left(\sqrt{-g} g^{\alpha\beta}\right)_{,\alpha} = 0. \tag{10}$$

This condition, as we will see further on, defines the coordinate system called harmonic.

And so, if it is always necessary to fix the coordinate system so that Einstein's equations form a closed system, then with this the restriction, commonly imposed upon the quantities defining the conservating characteristics for gravitational field, is eliminated, that is, that these are represented by tensorial quantities. For this reason it is natural that the density of the conservating quantities is described through non-tensorial objects, which depend on the coordinate system where they are given. The question about the physical meaning of each coordinate system and which of them to consider privileged, is partially resolved for insular systems in the framework of the problem of determination of the global dynamic characteristics, since for these systems it is natural to choose coordinates that behave in the infinite as a true Cartesian coordinates.

2. The Papapetrou's Pseudotensor

In 1948 Papapetrou [1] proposed for the conservation's laws a symmetrical variant of pseudotensor based on the study of the gravitational field related to the flat space-time, that is, in essence, with the bimetrical formalism, as it was shown clearly by Burlankov later [2].

Following Rosen [3,4], Papapetrou in his paper proposes to interpret the Einstenian gravitational theory, as the other field theories, relating to the background of the flat space-time, giving to the components of the metric tensor the sense of gravitational potential and introducing simultaneous and independently the metric tensor of the flat space-time. It should be noted that Papapetrou in his prior works about the conservation law of angular momentum in the GRT, arrived to the conclusion that this law should be formulated in the Einstenian theory only in the case of the explicit introduction of the flat world's metric tensor. This interpretation possesses the following mathematical advantage: all the quantities that are pseudotensors in the GTR, and all pseudotensorial relations, as for example the energy-momentum conservation law, acquire a true tensorial character if the covariant differentiation is introduced regarding a helping flat metric.

As it is known, in the special relativity theory the conservation law of angular momentum is consequence of two facts: in the first place the invariance of the action integral regarding the rotation transformations, and in the second place the property of symmetry of the matter energy-momentum tensor. In the GTR on the one hand it is not a natural notion of translation and rotation, and it does not permit to deduce the conservation law of the angular momentum directly from Noether's theorem, and on the other hand the expression obtained from this theorem for the complex energy-momentum does not possess the property of symmetry.

However, the introduction of the flat world's metric, that carries to a new transport independent from the trajectory, permits to apply in the GRT concepts such as translation and rotation, and therefore to formulate the conservation law for the angular momentum. From the fact that the noncovariant Lagrange function

$$\Lambda = \frac{\sqrt{-g}}{8\kappa} g^{\mu\nu} \left(\Gamma^{\alpha}_{\mu\nu} \Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\beta\mu} \Gamma^{\beta}_{\alpha\nu} \right) \qquad (11)$$

is a scalar density regarding linear transformations (translation and rotation), the energy-momentum conservation law can be obtained with the canonical pseudotensor for the gravitational field

$$\sqrt{-g} t^{\alpha}_{\beta} = \frac{\partial \Lambda}{\partial g_{\kappa\lambda,\alpha}} g_{\kappa\lambda,\beta} - \Lambda \delta^{\alpha}_{\beta}. \qquad (12)$$

To determine the rotation in the Riemannian space-time, we introduce the Galilean metric

$$\delta_{\mu\nu} = \delta^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1) \qquad (13)$$

and then the rotation transformation has the form

$$x'^{\mu} = x^{\mu} + \varepsilon \xi^{\mu}(x) \qquad |\varepsilon| \ll 1, \qquad (14)$$

where

$$\xi^{\mu} = \delta^{m\nu} \xi_{\nu} = \delta^{\mu\nu} \omega_{\nu\theta} r^{\theta} = \delta^{\mu\nu} \omega_{\nu\theta} (x^{\theta} - x^{\theta}_{0}) \qquad (15)$$

and x^{θ}_{0} is an arbitrary initial point, besides

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \qquad (16)$$

The invariance of the action function regarding transformations (14) gives the following identity:

$$\left\{ \sqrt{-g} \left[r^{\theta} \delta^{\mu\alpha} (T^{\nu}_{\alpha} + t^{\nu}_{\alpha}) - r^{\mu} \delta^{\theta\alpha} (T^{\nu}_{\alpha} + t^{\nu}_{\alpha}) + f^{\theta\mu\nu} \right] \right\}_{,\nu} = 0, \qquad (17)$$

where

$$\sqrt{-g} f^{\theta\mu\nu} = \frac{1}{2} \frac{\partial \Lambda}{\partial \gamma^{\alpha\beta}_{,\nu}} \left(\gamma^{\mu\alpha} \delta^{\theta\beta} - \gamma^{\theta\alpha} \delta^{\mu\beta} \right) \qquad (18)$$

and

$$\gamma^{\mu\nu} = \sqrt{-g} g^{\mu\nu}. \qquad (19)$$

The identity (17) expresses the conservation law for the angular momentum

$$F^{\theta\mu\nu} = \sqrt{-g} \left[r^{\theta} \delta^{\mu\alpha} (T^{\nu}_{\alpha} + t^{\nu}_{\alpha}) - r^{\mu} \delta^{\theta\alpha} (T^{\nu}_{\alpha} + t^{\nu}_{\alpha}) + f^{\theta\mu\nu} \right], \qquad (20)$$

the last term defines spin density of the field.

Another expression for the angular moment is obtained if we require that the energy-momentum complex be symmetrical. This way of definition was the one that Papapetrou utilized, employing the general method proposed by Belinfante [5]. Let the pseudotensor $\Theta^{\mu\nu}$ be a symmetrical energy-momentum density

$$\Theta^{\mu\nu} = \Theta^{\nu\mu}, \qquad (21)$$

that differs from the density $\sqrt{-g}\delta^{\mu\alpha}(T^{\nu}_{\alpha} + t^{\nu}_{\alpha})$ by the divergential term

$$\Theta^{\mu\nu} = \sqrt{-g} \delta^{\mu\alpha} (T^{\nu}_{\alpha} + t^{\nu}_{\alpha}) + B^{\mu\nu\alpha}_{,\alpha}. \qquad (22)$$

The equation of continuity is fulfilled if the quantity $B^{\mu\nu\alpha}$ is antisymmetric by the two last indices

$$B^{\mu\nu\alpha} = -B^{\mu\alpha\nu}. \qquad (23)$$

Papapetrou, in an analogous way as Belinfante did, found that $B^{\mu\nu\alpha}$ can be expressed with the aid of the spin density $f^{\mu\nu\alpha}$ by means of the relation

$$B^{\mu\nu\alpha} = -\frac{1}{2} \sqrt{-g} \left(f^{\mu\nu\alpha} + f^{\alpha\mu\nu} + f^{\alpha\nu\mu} \right). \qquad (24)$$

Substituting in (18) the Lagrangian density (11) we find the following expression

$$\Theta^{\mu\nu} = \frac{1}{2\kappa} \left(\gamma^{\mu\nu} \delta^{\alpha\beta} + \gamma^{\alpha\beta} \delta^{\mu\nu} - \gamma^{\mu\alpha} \delta^{\nu\beta} - \gamma^{\nu\beta} \delta^{\mu\alpha} \right)_{,\alpha\beta}. \qquad (25)$$

It is easy to verify from (22) and (24) that the corresponding angular momentum density

$$\Phi^{\theta\mu\nu} = r^{\theta} \Theta^{\mu\nu} - r^{\mu} \Theta^{\theta\nu} \qquad (26)$$

and the density $F^{\theta\mu\nu}$ (20), that is obtained from the invariant action function, are related through the formula

$$\Phi^{\theta\mu\nu} = F^{\theta\mu\nu} + \left(r^{\theta} B^{\mu\nu\alpha} - r^{\mu} B^{\theta\nu\alpha} \right)_{,\alpha}. \qquad (27)$$

From the structure of the relation (8) one can see that the De Donder-Fock coordinates condition

$$\gamma^{\mu\nu}_{,\nu} = 0 \qquad (28)$$

simplifies strongly the Papapetrou pseudotensor and carries it to the form

$$2\kappa \Theta^{\mu\nu} = -\overset{0}{\Box} \gamma^{\mu\nu}, \qquad (29)$$

where $\overset{0}{\Box} = \delta^{\mu\nu} \partial_{\mu} \partial_{\nu}$ is the D'Alambertian of the flat world. Einstein equations have this same form for the case of a weak gravitational field, with the only difference that on the left side is the energy-momentum tensor density of the matter $\sqrt{-g}T^{\mu\nu}$ instead of the pseudotensor $\Theta^{\mu\nu}$. Besides, this expression permits to carry out an analogy with the electrodynamics equations, where instead of the 4-vectors A_{μ} andj_{μ} we now have the quantities and correspondingly. On this base Gupta considered that the relation (29) expresses the equation for the determination of $\gamma^{\mu\nu}$, being the source for this field the total energy-momentum pseudotensor, that is, not only the matter creates gravitational field but the same gravitational field serves as a source of itself.

From this it follows, according to Gupta's opinion, the non-linearity of Einstein equations.

3. Determination of the Global Characteristics of Some Gravitational Fields

The solution of equation (29), as we all know, can be represented in the form

$$\gamma^{\mu\nu} = -\frac{2\kappa}{4\pi}\int\left(\frac{\Theta^{\mu\nu}}{R}\right)_{ret} dV', \tag{30}$$

where R is the radius built with the Cartesian coordinates regarding the metrica $\delta_{\mu\nu}$. In relation to this the idea emerges of utilizing the classical definitions of the global characteristics in the background of the flat space-time. For this the densities of these characteristics should be expressed through coordinates being harmonic regarding $g_{m\nu}$ and Cartesians regarding $\delta_{\mu\nu}$. The global characteristics will be obtained then by integration for all the tridimensional volume of the flat world, considering that the densities of the mass and of the total momentum are equal correspondingly to

$$\rho = \Theta^{00}, \qquad p^i = \Theta^{0i} \tag{31}$$

and the global characteristics are determined as

$$M = \int \Theta^{00} dV, \tag{32}$$

$$P^i = \int \Theta^{0i} dV, \tag{33}$$

$$L^i = \varepsilon_{ijg}\int \Theta^{0j} x^g dV, \tag{34}$$

$$D^i = \int \Theta^{00} x^i dV, \tag{35}$$

and

$$D^{ij} = \int \Theta^{00}\left(3x^i x^j - R^2\delta^i_j\right) dV, \tag{36}$$

where M is the total mass, P^i are the total momentum components, L^i are the total angular momentum components, D^i are the dipolar mass momentum components, D^{ij} are the quadrupolar mass momentum components, $R^2 = \sum\limits_{i=1}^{3} x^i x^i$, dV is the elementary tridimensional volume of flat world regarding $\delta_{\mu\nu}$ and the integration is carried out for all the space.

The choice of the Papapetrou pseudotensor in harmonic coordinates for the determination of the quantities (32)–(36) is favorable due to the fact that all these quantities permit, in the case of stationary fields, a transformation by means of the Gauss-Ostrodgaski theorem, to integration by bidimensional surfaces, where the subintegral expressions have a very simple form. In the stationary case of fields (29) takes the form $\Theta^{\mu\nu} = -\frac{1}{2\kappa}\gamma^{\mu\nu}_{,kk}$ and then

$$M = -\frac{1}{2k}\oint \gamma^{00}_{,k} dS^k, \tag{37}$$

$$P^i = -\frac{1}{2\kappa}\oint \gamma^{0i}_{,k} dS^k, \tag{38}$$

$$L^i = -\frac{\varepsilon_{ijg}}{2\kappa}\oint\left(\gamma^{0j}_{,k}x^g - \gamma^{0j}\delta^g_k\right) dS^k, \tag{39}$$

$$D^i = -\frac{1}{2\kappa}\oint\left(\gamma^{00}_{,k}x^i - \gamma^{00}\delta^i_k\right) dS^k, \tag{40}$$

$$D^{ij} = -\frac{1}{2\kappa}\oint\Big[\gamma^{00}_{,k}\left(3x^i x^j - R^2\delta^i_j\right) - \gamma^{00}\left(3x^i\delta^j_k + 3x^j\delta^i_k - 2x^k\delta^i_j\right)\Big] dS^k. \tag{41}$$

4. The Harmonic Coordinates

The harmonic coordinates were introduced by De Donder and Fock [6]. For the last one the physical sense of the harmonic coordinates is reduced to being like a generalization of the Cartesian coordinates of the flat world to the case of the curved space-time. This generalization contains two moments. In the first, in the flat world a inertial system of reference grants a privilege to the Cartesian coordinates due to the Lorentz transformations, expressing the properties of homogeneity and isotropy of the space-time, the mentioned transformations have a lineal form in these coordinates. In the case of the space-time homogeneous only in the infinite (the case of insular systems), it is possible to introduce a privileged coordinates system, to an accuracy of the Lorentz transformations. These coordinates, it is understood, should become true Cartesian for infinitely large distances from the source. In the second moment, as in the case of the Cartesian coordinates, the harmonic coordinates, in agreement with Fock, eliminate all the fictitious gravitational fields.

Now we will analyze the issue about the method for the determination of the harmonic coordinates and also about the possibility of obtaining them for a given gravitational field.

The method to determine the harmonic coordinates was given by Fock and consists in the following: suppose that there is a metric $g_{\mu\nu}(x^\alpha)$ in arbitrary non-harmonic coordinates $\{x^\alpha\}$; then from the De Donder-Fock condition

$$\frac{\partial}{\partial x^\mu_\Gamma}\left(\sqrt{-g_\Gamma}\, g^{\mu\nu}_\Gamma\right) = 0,$$

where $\{x^\mu_\Gamma\}$ are the harmonic coordinates, g_Γ and $g^{\mu\nu}_\Gamma$ are the metric determinant and contravariant metric tensor in these coordinates, it follows that

$$\Box x^\mu_\Gamma = 0, \tag{42}$$

where $\Box \equiv g^{\mu\nu}()_{;\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)$ is the D'Alambert operator in the initial coordinates $\{x^\mu\}$.

This means that as harmonic coordinates any four independent solutions of the D' Alambert equation

$$\Box\chi = 0, \tag{43}$$

can be taken, which, however, should possess the corresponding asymptotic properties, just at infinitely large distances from the source (since it is assumed the source is found in a limited region of the space) they should be transformed into the habitual Cartesian coordinates. In view of the linearity of the D' Alambert equation it is easy to see that these solutions are determined to an accuracy of the Lorentz transformations. This property and the fact that these coordinates are asymptotically flat, permit to conclude that all the fictitious gravitational fields are automatically eliminated in the reference system described by these coordinates.

Finally, we will analyze the matter about when it is possible to solve the D' Alambert equation (43) by the separation of variables method and what conditions should be complied with for this. We will limit ourselves to analyze only the physical side of the problem, which is the searching for solutions behaving in the infinite as true Cartesian coordinates for stationary fields with axial symmetry.

As it is known, in the case of stationary and axial symmetric metric two Killing vectors exist

$$\underline{t} = \partial_t \quad \text{and} \quad \underline{\varphi} = \partial_\varphi. \tag{44}$$

For this reason, the components of the metric tensor will contain as variables only the "essential"coordinates r and θ, and the general form of the metric will be given as

$$ds^2 = g_{tt}(r,\theta)dt^2 + g_{rr}(r,\theta)dr^2 + 2g_{t\varphi}(r,\theta)dtd\varphi + g_{\theta\theta}(r,\theta)d\theta^2 + g_{\varphi\varphi}(r,\theta)d\varphi^2. \tag{45}$$

It can be demonstrated that the D'Alambert equation (43) permits separation of variables with solutions

$$\chi = e^{in\varphi}R(r)\Phi(\theta), \tag{46}$$

if the following components of the metric tensor density have the form

$$\begin{aligned} \gamma^{rr}(r,\theta) &= f_1(r)g_1(\theta), \\ \gamma^{\theta\theta}(r,\theta) &= f_2(r)g_2(\theta), \\ \gamma^{\varphi\varphi}(r,\theta) &= g_1(\theta)F(r) + f_2(r)G(\theta). \end{aligned} \tag{47}$$

The functions $R(r)$ and $\Phi(\theta)$ satisfy the following ordinary differential equations

$$\frac{d}{dr}\left[f_1(r)\frac{dR}{dr}\right] - [n^2F(r) + Af_2(r)]R = 0, \tag{48}$$

$$\frac{d}{d\theta}\left[g_2(\theta)\frac{d\Phi}{d\theta}\right] - [n^2G(\theta) - Ag_1(\theta)]\Phi = 0, \tag{49}$$

where A and n are constants that should be chosen in a way that the assembly of the three solutions become Cartesian coordinates to infinitely large distances r:

$$\{\chi\} \overset{r\to\infty}{\longrightarrow} \{r\sin\theta\cos\varphi, r\sin\theta\cos\varphi, r\cos\varphi\}. \tag{50}$$

In the choosing of the spatial solutions (41) the dependence of the temporary coordinate t is excluded because, as is easy to verify, this coordinate t is harmonic and therefore, from considerations about their independence and the stationary character of the gravitational field, the other coordinates should not contain this variable.

5. The Schwarzschild field in harmonic coordinates

The solution obtained by Schwarzschild [7] defines the gravitational field created by any mass distribution with spherical symmetry in the vacuum. This solution was obtained by Fock in harmonic coordinates [6] by resolving Einstein equations considering the De Donder-Fock condition (10). We obtained it employing the method described in the previous section. For this we start from the Schwarzschild metric written in curvature coordinates

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2\left(d\theta^2 + \sin\theta d\varphi^2\right) \tag{51}$$

with the determinant of the tensor metric equals to $g = -r^4\sin^2\theta$, and

$$\sqrt{-g} = r^2\sin\theta. \tag{52}$$

The components of the contravariant metric tensor density are

$$\begin{aligned} \gamma^{tt} &= \frac{r^3\sin\theta}{r-2m}, \\ \gamma^{rr} &= -r(r-2m)\sin\theta, \\ \gamma^{\theta\theta} &= -\sin\theta, \\ \gamma^{\varphi\varphi} &= -\frac{1}{\sin\theta} \end{aligned} \tag{53}$$

comparing (53) with (47) we find

$$\begin{aligned} g_1(\theta) &= -\sin\theta, & f_1(r) &= r(r-2m), \\ g_2(\theta) &= -\sin\theta, & f_2(r) &= 1, \\ G(\theta) &= -\frac{1}{\sin\theta}, & F(r) &= 0. \end{aligned} \tag{54}$$

Then the D'Alambert equation separates in to

$$\frac{d}{dr}\left[r(r-2m)\frac{dR}{dr}\right] - AR = 0, \tag{55}$$

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Phi}{d\theta}\right] - \left[\frac{n^2}{\sin\theta} - A\sin\theta\right]\Phi = 0, \tag{56}$$

and the harmonic coordinates are determined by the formula (46).

The equation (56) has solution only when

$$A = l(l+1), \tag{57}$$

and in this case solutions are Legendre associated polynomials of first type with argument $\cos\theta$

$$\Phi_{\ln} = P_l^n(\cos\theta), \tag{58}$$

besides

$$P_l^n(x) = \frac{(1-x^2)^{n/2}}{2^l l!}\frac{d^{l+n}}{dx^{l+n}}(x^2-1)^l,$$
$$l = 0,1,2,\dot{;}\, n = 0,1,2,\ldots,l. \tag{59}$$

The asymptotic form of the solution (50) says we should take $l = 1$, $n = 0,1$. Then the equation (55) becomes

$$\frac{d}{dr}\left[r(r-2m)\frac{dR}{dr}\right] - 2R = 0, \tag{60}$$

from here one can obtain $R = r - m$ as the only solution that grows when r monotonous increases. Finally we obtain for $l = 1$, $n = 1$

$$\chi_1 = x_\Gamma = (r-m)\sin\theta\cos\varphi, \tag{61}$$
$$\chi_2 = y_\Gamma = (r-m)\sin\theta\sin\varphi, \tag{62}$$

and for $l = 1$, $n = 0$

$$\chi_3 = z_\Gamma = (r-m)\cos\varphi, \tag{63}$$
$$\chi_0 = t_\Gamma = t. \tag{64}$$

These are the appropriate harmonic coordinates for the Schwarzschild field behaving as true Cartesian coordinates when $r \to \infty$.

The Jacobian of the transformation is equal to $J = (r-m)^2\sin\theta$.

Note that from (61)–(63) the spherical coordinates corresponding to the Cartesian harmonic ones are

$$R = r - m, \theta, \varphi \tag{65}$$

and

$$\sqrt{-g_\Gamma} = \frac{(R+m)^2}{R^2}. \tag{66}$$

Therefore the Schwarzschild metric in harmonic spherical coordinates is written as

$$ds^2 = \frac{R-m}{R+m}dt^2 - \frac{R+m}{R-m}dr^2$$
$$- (R+m)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{67}$$

But for our calculations we need the components of the metric density in Cartesian harmonic coordinates. We will only show those that will be necessary for us

$$\gamma_\Gamma^{tt} = \frac{(R+m)^3}{R^2(R-m)}, \quad \gamma_\Gamma^{tx} = \gamma_\Gamma^{ty} = \gamma_\Gamma^{tz} = 0 \tag{68}$$

and at once we will show the development of γ_Γ^{tt} in powers of the inverse of R

$$\gamma_\Gamma^{tt} = 1 + \frac{4m}{R} + \frac{7m^2}{R^2} + \frac{8m^3}{R^3} + \ldots \tag{69}$$

We calculate now the mass of the Schwarzschild system by the formula (32), taking as bidimensional surface the surface of the sphere with radius R

$$dS^k = x_\Gamma^k R\sin\theta d\theta d\varphi. \tag{70}$$

Then the mass is given for the expression

$$M = -\frac{1}{2\kappa}\oint x_\Gamma^k \gamma_{\Gamma,k}^{tt} R\sin\theta d\theta d\varphi. \tag{71}$$

Considering that $x_\Gamma^k \frac{\partial}{\partial x_\Gamma^k} = R\frac{\partial}{\partial R}$ we obtain

$$M = \frac{1}{2\kappa}\oint\left(4m + 14\frac{m^2}{R^2} + \ldots\right)\sin\theta d\theta d\varphi. \tag{72}$$

If we enlarge the radio of the sphere, we obtain in this limit the total mass content in all the space

$$M = m \tag{73}$$

In the same way, employing the formulae (38)–(41), the other characteristics of the source of the Schwarzschild field are obtained. The dipolar momentum results to be equal to zero, for which the center of mass is found in the point $R = 0$. The other characteristics also have null value, just as it should be expected from the physical interpretation of the Schwarzschild field.

6. Kerr field in harmonic coordinates

The first solution of the Einstein equations describing a stationary gravitational field with axial symmetry was discovered by Kerr [8]. Its aspect in Boyer-Lindquist coordinates is

$$ds^2 = \left(1 - \frac{2mr}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta^2}dr^2$$
$$+ \frac{4mar}{\rho^2}\sin^2\theta d\varphi dt - \rho^2 d\theta^2$$
$$- \left(r^2 + a^2 + \frac{2mra^2}{\rho^2}\right)\sin^2\theta d\varphi^2, \tag{74}$$

where the notations

$$\Delta^2 = r^2 - 2mr + a^2,$$
$$\rho^2 = r^2 + a^2\cos\theta \tag{75}$$

have been introduced. For $a = 0$ the Kerr metric becomes the Schwarzschild metric (51). By comparing the Kerr metric for small a with the approximate

Lense-Tirring metric characterizing the metric around a slowly rotating body with an angular momentum L [9]

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \frac{4L\sin^2\theta}{r} dt d\varphi$$

it is deduced $a = L/m$. This relation can also be verified if the total angular momentum is calculated by means of the method proposed in section 3, which we will calculate further on.

From (74) we find

$$\sqrt{-g} = \rho^2 \sin\theta. \quad (76)$$

Continuing, we will present the contravariant components of $g^{\mu\nu}$, carrying them to the following expression for the 4-gradient operator

$$g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{\Delta^2}\left(r^2 + a^2 + \frac{2mra^2}{\rho^2}\right)\partial_t^2 - \frac{\Delta^2}{\rho^2}\partial_r^2 - \frac{1}{\rho^2}\partial_\theta^2 - \frac{1}{\Delta^2\sin^2\theta}\left(1 - \frac{2mr}{\rho^2}\right)\partial_\varphi^2 + \frac{2mar}{\rho^2\Delta^2}\partial_\varphi\partial_t, \quad (77)$$

and the components of the density of the contravariant metric tensor are equal to

$$\begin{aligned} \gamma^{tt} &= \frac{\rho^2}{\Delta^2}\left(r^2 + a^2 + \frac{2mra^2}{\rho^2}\sin^2\theta\right)\sin\theta, \\ \gamma^{t\varphi} &= \frac{mar}{\Delta^2}\sin\theta, \quad \gamma^{\theta\theta} = -\sin\theta, \\ \gamma^{rr} &= -\Delta^2\sin\theta, \quad \gamma^{\varphi\varphi} = -\frac{1}{\sin\theta} + \frac{a^2}{\Delta^2}\sin\theta, \end{aligned} \quad (78)$$

then from (47) we find

$$\begin{aligned} g_1(\theta) &= -\sin\theta, \quad f_1(r) = \Delta^2, \\ g_2(\theta) &= -\sin\theta, \quad f_2(r) = 1, \\ G(\theta) &= -\frac{1}{\sin\theta}, \quad F(r) = -\frac{a^2}{\Delta^2}. \end{aligned} \quad (79)$$

The D'Alambert equation for determining the harmonic coordinates is separated in

$$\frac{d}{dr}\left[(r^2 - 2mr + a^2)\frac{dR}{dr}\right] + \left[\frac{n^2a^2}{r^2 - 2mr + a^2} - A\right] R = 0, \quad (80)$$

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Phi}{d\theta}\right] - \left[\frac{n^2}{\sin\theta} - A\sin\theta\right]\Phi = 0, \quad (81)$$

and the harmonic coordinates are determined by the formula (46).

The equation (81) coincides with the equation (56) and, as it was indicated for the case of the Schwarzschild field, it makes sense to take only the values $l = 1$ ($A = 2$) and $n = 0, 1$. The case $n = 0$ gives again the equation (60) and therefore the harmonic coordinate z_Γ is equal to

$$z_\Gamma = (r - m)\cos\theta,$$

as in the Schwarzschild field. This coincidence is explained in part due to z_Γ being the axis of symmetry of the Kerr field.

For $l = 1$, $n = 1$ the equation (80) becomes

$$\frac{d}{dr}\left[(r^2 - 2mr + a^2)\frac{dR}{dr}\right] + \left[\frac{a^2}{r^2 - 2mr + a^2} - 2\right] R = 0. \quad (82)$$

The equation (82) has different solutions in dependence of the relation between a^2 and m^2 . However, physically the relation $a^2 \leq m^2$ is only permitted, since in contrary case ($a^2 > m^2$) a not physical situation appears: the causality principle is violated, that is, closed time-like world lines appear.

In this work the following exact solutions were found:

I. I. For $a^2 < m^2$

$$R_1(r) = P(r)\big|_{a^2<m^2} = (r - m)\cos A\xi - a\sin A\xi \quad (83)$$

where $\xi = \frac{1}{2}\ln\left|\frac{r - m - \sqrt{m^2 - a^2}}{r - m + \sqrt{m^2 - a^2}}\right|$ and $A^2 = \frac{a^2}{m^2 - a^2}$.

II. For $a^2 = m^2$

$$R_2(r) = P(r)\big|_{a^2=m^2} = (r - m)\cos\frac{a}{r - m} + a\sin\frac{a}{r - m}. \quad (84)$$

Note that only the solutions (83) and (84) behave in the infinite as the Cartesian radius r. Besides, for $a_1 = 0$, R_1 becomes the associated-with-the-harmonic-coordinates radius for the Schwarzschild field (R_2 is simply converted in the radius of the flat world)

$$R_1\big|_{a=0} = r - m \quad \text{and} \quad R_2\big|_{a=0} = r. \quad (85)$$

It could be questioned that the solutions (83) and (84) depend on the radial coordinate through trigonometric functions. However, for our purpose this situation does not have singular importance, due to our use of the characteristics of the fields only at distances r sufficiently large. It can be shown, if the functions (83) and (84) are analyzed in detail, that they grow monotonous upon growing r in the regions where the variable r is greater than certain critical value. These critical values are:

$$r_{1\,crit} = m + \frac{a}{A}\,\mathrm{cth}\,\frac{\pi}{2A} \quad \text{for} \quad a^2 < m^2 \tag{86}$$

and

$$r_{2\,crit} = m + \frac{2a}{\pi} \quad \text{for} \quad a^2 = m^2. \tag{87}$$

It is notable that the solutions R_1 and R_2 for large values of r have the same series development in with an accuracy to the terms of fifth order of smallness, that is

$$P(r) = r - m + \frac{a^2}{2r} + \frac{a^2 m}{2r^2} + \frac{a^2}{8r^3}(4m^2 - a^2) + \frac{a^2 m}{8r^4}(4m^2 - 3a^2) + O(r^{-5}), \tag{88}$$

for this reason from now on we will not distinguish these two cases.

As it was said previously, in spite of the case $m^2 < a^2$ not having physical interest it will still be considered with the solution of equation (82). Since, and besides, as the calculations show, this solution behaves in the infinite as the Cartesian radius r, satisfies the condition $R_3\big|_{a=0} = r - m$, and, what turns out to be interesting, the development in series of powers in R coincides with (88). In this manner, this solution possesses the form

$$R_3(r) = P(r)\big|_{a^2>m^2} = (r-m)\cos B\left(\eta - \frac{\pi}{2}\right) - a\sin B\left(\eta - \frac{\pi}{2}\right), \tag{89}$$

where $\eta = \mathrm{arctg}\,\dfrac{r-m}{\sqrt{a^2-m^2}}$ and $B^2 = \dfrac{a^2}{a^2-m^2}$.

Solution (89) also grows monotonous upon growing r in the regions where the variable r is greater than the critical value

$$r_{3\,crit} = m + \frac{a}{B}\,\mathrm{ctg}\,\frac{\pi}{2B} \quad \text{for} \quad a^2 > m^2. \tag{90}$$

The fact that the three solutions (83),(84) and (89) have the same asymptotic behavior with an accuracy to the terms of fifth order of smallness, tells that these cases are undistinguishable from the point of view of the definition of the some global dynamics characteristic of this field. However, as it was expected, the values of the multipolar moments are different starting from certain order due to its calculation being necessary to consider more terms in the development of the function $R(r)$.

Thus, finally we obtained the harmonic coordinates for the Kerr metric

$$\begin{aligned} x_\Gamma &= P(r)\cos\varphi\sin\theta, \\ y_\Gamma &= P(r)\sin\varphi\sin\theta, \\ z_\Gamma &= (r-m)\cos\theta, \\ t_\Gamma &= t, \end{aligned} \tag{91}$$

where the function $P(r)$ can be written in the exact form (83),(84) or (89) in dependence of the relation between a^2 and m^2, or by means of the asymptotic development (88) for the all cases.

7. Dynamical Characteristics of Kerr Field

In calculating global dynamic characteristics for Kerr field by the method described in Section III, we for convenience, introduce the spherical coordinates corresponding to the harmonic ones (91) by the formulae

$$\begin{aligned} x_\Gamma &= R_\Gamma \cos\varphi_\Gamma \sin\theta_\Gamma, \\ y_\Gamma &= R_\Gamma \sin\varphi_\Gamma \sin\theta_\Gamma, \\ z_\Gamma &= R_\Gamma \cos\theta_\Gamma. \end{aligned} \tag{92}$$

The inverse transformation has the form

$$\begin{aligned} R_\Gamma &= \sqrt{x_\Gamma^2 + y_\Gamma^2 + z_\Gamma^2}, \\ \varphi_\Gamma &= \mathrm{arctg}\,\frac{y_\Gamma}{x_\Gamma}, \\ \theta_\Gamma &= \arccos\frac{z_\Gamma}{\sqrt{x_\Gamma^2 + y_\Gamma^2 + z_\Gamma^2}}. \end{aligned} \tag{93}$$

Then from (93),(91) and (88) we find the link between harmonic spherical coordinates $(R_\Gamma, \theta_\Gamma, \varphi_\Gamma)$ and Boyer-Lindquist spherical coordinates (r, θ, φ):

$$R_\Gamma = r - m + \frac{a^2\sin^2\theta}{2r} + \frac{a^2 m\sin^2\theta}{2r^2} + \frac{a^2\sin^2\theta}{4r^3}(2m^2 - a^2\sin^2\theta) + \frac{a^2 m\sin^2\theta}{4r^4}(2m^2 - 3a^2\sin^2\theta) + O(r^{-5}), \tag{94}$$

$$\cos\theta_\Gamma = \cos\theta\Big[1 - \frac{a^2\sin^2\theta}{2r^2} - \frac{a^2 m\sin^2\theta}{r^3} - \frac{a^2\sin^2\theta}{2r^4}(3m^2 - a^2\sin^2\theta) - \frac{2a^2 m\sin^2\theta}{r^5}(m^2 - a^2\sin^2\theta) + O(r^{-6})\Big], \tag{95}$$

$$\varphi_\Gamma = \varphi. \tag{96}$$

Spherical coordinates (93) are introduced only for convenience in calculations because, as it is clear from Section III, all quantities in formulae (37)–(41) and also the surface element dS^k must be written in "Cartesian"coordinates (91). However since we choose as surface of integration spherical ones then it is convenient to express all quantities by harmonic coordinates through correspondent spherical (93) as well as the spherical surface element

$$dS^k = x_\Gamma^k R_\Gamma \sin\theta_\Gamma d\theta_\Gamma d\varphi_\Gamma. \tag{97}$$

The Jacobian of this transformation $J = \det\left\|\dfrac{\partial x_\Gamma}{\partial \chi}\right\|$, where $\{x_\Gamma\} = \{t_\Gamma, x_\Gamma, y_\Gamma, z_\Gamma\}$ and $\{\chi\} = \{t, r, \theta, \varphi\}$, is

$$J = PQ\sin\theta, \tag{98}$$

where

$$Q = (r-m)\frac{dP}{dr}\sin^2\theta + P\cos^2\theta, \tag{99}$$

besides

$$\begin{aligned} Q = r - m + \frac{a^2\cos 2\theta}{2r} + \frac{a^2 m\cos 2\theta}{2r^2} \\ + \frac{a^2}{4r^3}\left[4m^2\cos 2\theta + a^2(3-4\cos^2\theta)\right] \\ + \frac{a^2 m}{8r^4}\left[4m^2\cos 2\theta + 3a^2(3-4\cos^2\theta)\right] \\ + O(r^{-5}). \end{aligned} \tag{100}$$

From (76) and (98) we find determinant of metric tensor in harmonic coordinates and then

$$\sqrt{-g_\Gamma} = |J|^{-1}\sqrt{-g} = \frac{\rho^2}{PQ}. \tag{101}$$

From (37) and (97) we write the expression for mass inside volume limited by sphere with radius R_Γ:

$$M = -\frac{1}{2\kappa}\oint x^k_\Gamma \gamma^{00}_{\Gamma,k} R_\Gamma \sin\theta_\Gamma d\theta_\Gamma d\varphi_\Gamma. \tag{102}$$

Note that $g^{00}_\Gamma = g^{00}$ and for this reason

$$\gamma^{00}_\Gamma = \frac{(r^2+a^2)^2 - a^2\Delta^2\sin^2\theta}{PQ\Delta^2}, \tag{103}$$

and the correspondent development has the form

$$\begin{aligned} \gamma^{00}_\Gamma = 1 + \frac{4m}{r} + \frac{11m^2}{r^2} + \frac{2m}{r^3}(13m^2 - a^2\cos^2\theta) \\ + \frac{m^2}{r^4}\left[57m^2 + (4-5\cos^2\theta)a^2\right] + O(r^{-5}). \end{aligned} \tag{104}$$

The following relations hold

$$\begin{aligned} x^k_\Gamma\frac{\partial}{\partial x^k_\Gamma} = \frac{1}{Q}\Bigg[P(r-m)\frac{\partial}{\partial r} \\ + \left(P - (r-m)\frac{dP}{dr}\right)\sin\theta\cos\theta\frac{\partial}{\partial\theta}\Bigg], \end{aligned} \tag{105}$$

$$\begin{aligned} P - (r-m)\frac{dP}{dr} = \frac{a^2}{r} + \frac{a^2 m}{r^2} + \frac{a^2}{2r^3}(2m^2 - a^2) \\ + \frac{ma^2}{4r^4}(4m^2 - 3a^2) + O(r^{-5}). \end{aligned} \tag{106}$$

From (104)–(106) we have

$$\begin{aligned} x^k_\Gamma\gamma^{00}_{\Gamma,k} = -\frac{4m}{r} - \frac{18m^2}{r^2} - \frac{m}{r^3}(56m^2 - 2a^2\cos\theta) \\ - \frac{m^2}{r^4}\left[150m^2 + (40 - 32\cos^2\theta)a^2\right] \\ + O(r^{-5}). \end{aligned} \tag{107}$$

For the calculation of the mass it is enough to take only the first terms in the development of the quantities that enter in the integral (102) in the approximation for (94) as $r = R_\Gamma + m$ (the other terms are necessary to calculate the dipolar and quadrupolar mass moments):

$$x^k_\Gamma\gamma^{00}_{\Gamma,k} = -\frac{4m}{r} + O(r^{-2}) = -\frac{4m}{R_\Gamma} + O(R_\Gamma^{-2}), \tag{108}$$

$$\sin\theta_\Gamma d\theta_\Gamma = \left[\sin\theta + O(R_\Gamma^{-1})\right]d\theta, \quad d\varphi_\Gamma = d\varphi. \tag{109}$$

Replacing (108) and (109) in (102) we obtain

$$M = -\frac{1}{2\kappa}\oint\left[-4m + O(R_\Gamma^{-1}\right]\sin\theta d\theta d\varphi. \tag{110}$$

When R_Γ tends to infinity we find the total mass

$$M = m. \tag{111}$$

In the same way, employing the formulae (38)-(41), the other global characteristics of the source of the Kerr field are obtained. We will not reproduce in detail these laborious calculations and we will limit ourselves to enunciate the main results.

Our calculations show that the total moment components are zero

$$P^x = P^y = P^z = 0, \tag{112}$$

which indicates that the system is in rest; the total angular momentum components are

$$L^x = L^y = 0 \quad \text{and} \quad L^z = am, \tag{113}$$

there is in agreement with the physical interpretation of the parameters a and b obtained from other considerations (see for example the approximate Lense-Tirring metric characterizing the metric around a slowly rotating body with an angular momentum $L = am$ [9]); the dipolar mass momentum components result to be equal to zero

$$D^x = D^y = D^z = 0, \tag{114}$$

for which the center of mass is found in the point $R_\Gamma = 0$; and finally the quadrupolar mass momentum components are equal to

$$\begin{aligned} D^{xx} = D^{yy} = \frac{2}{5}a^2 m, \\ D^{zz} = -\frac{4}{5}a^2 m, \\ D^{xy} = D^{yz} = D^{zx} = 0. \end{aligned} \tag{115}$$

It is interesting to observe that the values for the quadrupolar mass moment components (114) and (115) coincide with those that are obtained in classic mechanics when the mass quadrupolar moment is calculated for an ellipsoid of revolution with constant mass density and where the parameter m is the total mass of the solid while the parameter a^2 characterizes the difference between the squares of the major and minor semi axis.

8. Conclusion.

In the present work a method for calculation of the global dynamic characteristics for insular and stationary gravitational fields with axial symmetry using as building material the Papapetrou's pseudotensor has been developed. For this definition of the energy-momentum density the harmonic coordinates turn out to be mathematical and physically appropriate for the development of this problem. In fact, in this way the global characteristics definition can extend from the special relativity theory to the general one for the case of a curved space-time, which then can be described by the metric tensor field determined by the equation (29) in the background of a flat space-time.

The form of the metric, which allows variable separation in the D'Alambert equation and therefore also allows finding the corresponding harmonic coordinates, has been obtained. In quality of application examples, we have carried out the calculations to obtain the harmonic coordinates and the global dynamic characteristics for the Schwarzschild and Kerr fields. The obtained results for them confirm, in principle, the physical interpretation commonly associated to these systems [10–13]. One can apply this method for the interpretation of other insular stationary solutions with axial symmetry of the Einstein equations such as the solutions NUT [14] and pencil of light [15].

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