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On the compatibility of irreversible dynamics with the special theory of relativity

Contents

Abstract

The irreversible extensions of dynamics of Poincaré non-integrable systems are compatible with relativistic theory. This is shown by extending the Lorentz transformation to decaying states in a relativistic model of interacting fields. The non-local action is defined beyond the Hilbert space.

1. Introduction

Using the opportunity to contribute to this special volume devoted to the 100 years of the Special Theory of Relativity (STR) I would like to review recent ideas [1] developed under the guidance of Ilya Prigogine who always insisted that describing irreversible phenomena one can account for relativistic effects and wise versa, the relativistic description can encompass irreversible dynamics. This goes to the long standing question of reconciliation of the deterministic, time-reversible fundamental dynamical laws of physics with irreversible phenomena which were always thought to be incompatible with the dynamical laws. Since Boltzmann's time the transition from time-reversible evolution to irreversible kinetic theory was believed to require approximations. Therefore irreversible evolution becomes a result of the loss of information about the system and therefore, less precise, not fundamental, and moreover, "antropomorphic", a result of human inability to account for all details of the full time-reversible evolution. In contrary, Prigogine insisted on the objective and fundamental character of irreversible evolution, and therefore, on the possibility to derive mathematically rigorously the irreversible dynamical laws with broken time symmetry from time-reversible dynamics. Indeed, the recently discussed [3]-[10] origin of irreversible behaviour due to resonances in various Poincaré non-integrable quantum systems has opened a possibility to replace traditional approximations like coarse graining by extensions of dynamics outside the Hilbert space with new solutions describing unstable states evolving irreversibly. Thus, it was shown that the presence of resonances leads to intrinsic irreversibility of dynamical systems. Of course, as always, there is a price to pay. Such states defined in the wave function space and usually called Gamow vectors [11, 12] are not physical states in the conventional sense as they cannot be represented by positive bounded operators because they belong to extended functional spaces. However, their scalar products with suitable test functions give meaningful amplitudes and probabilities. A similar example may be the Wigner function, which is not a probability distribution in usual sense, but it provides nevertheless correct average values like usual probability distributions. In this sense, irreversible evolution is compatible with fundamental time-reversible dynamical laws.

Following this idea, we have shown [1] that one can introduce relativistic invariance into irreversible description. Ilya Prigogine considered this question as very important, in view of recent publications on this topic, e.g. [2], which show that adopting the determinism of STR and of the General Theory of Relativity one arrives at a view of the universe as a lifeless "space-time block" [2] in which there is no place for probability, chance and the difference between past and future is formal. Strongly opposing this point of view Ilya Prigogine wanted to show explicitly that relativistic theory is compatible with irreversible dynamics.

In order to demonstrate this compatibility we studied relativistic transformations of decaying unstable Gamow modes using a relativistic invariant model of two interacting fields introduced in [4]. We suggest that a proper relativistic transformation of the Gamow states is induced by the Lorentz transformation of field modes which are eigenstates of the 4-dimensional energy-momentum of the system. We begin with the formal application of the Lorentz boost to the Gamow state and show that this leads to a complex value of the transformed momentum $\mathbf{k}'$ [4]. However, we demonstrate that one can keep real the transformed momentum by considering a transformation at the level of wave packets. Our arguments are based on the fact that the Gamow state $|\Psi^{\mathrm{G}}(\mathbf{k})\rangle$ with momentum $\mathbf{k}$ is obtained as pole contribution from the incoming eigenstate of the energy-momentum $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$ using analytic continuation to the complex plane of the energy E for fixed momentum $\mathbf{k}$. Wave packets observed experimentally in scattering experiments are superpositions of the energy-momentum eigenstates. The analytic extension to the complex energy plane isolates the contribution of the exponentially decaying Gamow modes to the wave packet. All these Gamow modes have real momenta. The same wave packets will be seen by some moving observer as transformed by the proper Lorentz boost. The corresponding analytic extension for the transformed wave packet will isolate other Gamow modes with again real momenta thus determining the relativistic transformation of the Gamow modes preserving the reality of the momentum. We consider this construction as quite natural because from our point of view the Gamow modes are not independent entities but those contributions to the wave packet that obey purely exponential decay laws. The novelty of this approach is the non-point-wise implementable Lorentz boost in the extended complex energy-momentum space. Another approach including a representation of the Poincaré group acting on the Gamow states can be found in [13, 14].

Following closely [1] I present in section 2 the model and the exact solution obtained in [4]. The decaying Gamow modes obtained by analytic extension of this solution are introduced in section 3. The non-pointwise Lorentz transformation of exponentially decaying states is presented and discussed in section 4. Section 5 concludes the paper.

2. Exact solution for a model system

We consider a model system studied in [4] which includes two interacting quantum relativistic boson fields $\phi(x)$ and $\psi(x,q)$ in 4-dimensional Minkowski space with the metrics (+,-,-,-) so that the scalar product of 4-dimensional vectors $x \equiv (t,\mathbf{x})$ is given by $(x,x') \equiv tt' - (\mathbf{x}\mathbf{x}')$. A particular property of this system, which makes it useful for our purpose, is that whereas the first field $\phi(x)$ is a usual local field parametrized by the 4-dimensional vector x, the second field $\psi(x,q)$ is bilocal because it has an additional degree of freedom parametreized by a one-dimensional variable q. This additional degree of freedom will be the continuum, which can resonate leading to the decay of each momentum mode of the local field.

In the units where the speed of light is $\hbar = c = 1$, our local field is given by

$$\phi(x)\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})} \times \left[a^\dagger(\mathbf{k})e^{i(\omega_\mathbf{k}t-\mathbf{k}\mathbf{x})} + a(\mathbf{k})e^{-i(\omega_\mathbf{k}t-\mathbf{k}\mathbf{x})}\right],$$

$$\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2+\mathrm{M}^2}. \tag{1}$$

The energy (frequency) of the field modes $\omega(\mathbf{k})$ is determined by the momentum $\mathbf{k}$ and the mass of the particle M. The creation $a^\dagger(\mathbf{k})$ and annihilation $a(\mathbf{k})$ operators of the field modes satisfy the commutation relation for bosons:

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = (2\pi)^3 2\omega(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'). \tag{2}$$

For our convenience we require that the bilocal scalar field $\psi(x,q)$ is even in the internal degree of freedom q satisfying $\psi(x,q) = \psi(x,-q)$ and is given by

$$\psi(x,q) = \int \mathrm{d}^3\mathbf{k} \int_{E_0(\mathbf{k})}^{\infty} \mathrm{d}E \frac{\cos\left(\kappa(E,\mathbf{k})q\right)}{(2\pi)^4\kappa(E,\mathbf{k})} \times \left[b^\dagger(E,\mathbf{k})e^{i(\omega_\mathbf{k}t-\mathbf{k}\mathbf{x})} + b(E,\mathbf{k})e^{-i(\omega_\mathbf{k}t-\mathbf{k}\mathbf{x})}\right], \tag{3}$$

where

$$\kappa(E,\mathbf{k}) = (E^2-\mathbf{k}^2-4m^2)^{1/2},$$
$$E_0(\mathbf{k}) = (4m^2+\mathbf{k}^2)^{1/2}, \tag{4}$$

and m is the mass of the particle. The creation $b^\dagger(E,\mathbf{k})$ and annihilation $b(E,\mathbf{k})$ operators of the field modes satisfy the commutation relation for bosons

$$\left[b(E,\mathbf{k}), b^\dagger(E',\mathbf{k}')\right] = (2\pi)^4 k(E,\mathbf{k})\delta(E-E')\delta^3(\mathbf{k}-\mathbf{k}'). \tag{5}$$

The commutation relations (2) and (5) induce an algebra of the Poincare group generators. Free fields $\phi(x)$,

$\psi(x,q)$ are scalars, which are invariant with respect to the Poincaré group.

We introduce a simple quadratic interaction, which ensures that the whole theory keeps this invariance. Then the Hamiltonian is the zero component of the 4-momentum [4], [6]

$$P_0 = H = H_M + H_m + \lambda V \tag{6}$$

where the free Hamiltonian of the scalar field is

$$H_{\mathrm{M}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})}\,\omega(\mathbf{k})\,a^\dagger(\mathbf{k})a(\mathbf{k}), \tag{7}$$

the free Hamiltonian of the bilocal field is

$$H_m = \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E}{(2\pi)^4 k(E,\mathbf{k})} E\, b^\dagger(E,\mathbf{k}) b(E,\mathbf{k}), \tag{8}$$

and the quadratic interaction is

$$V = \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E\alpha(k(E,\mathbf{k}))}{(2\pi)^3 2\omega(\mathbf{k}) k(E,\mathbf{k})} \times \left(a^\dagger(\mathbf{k}) + a(-\mathbf{k})\right)\left(b^\dagger(E,\mathbf{k}) + b(E,-\mathbf{k})\right) \tag{9}$$

where $\alpha(x)$ is a scalar function with good asymptotic behaviour, which allows us to avoid divergence.

The generator of three-dimensional translations

$$\mathbf{P} = \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E}{(2\pi)^4 k(E,\mathbf{k})}\, \mathbf{k}\, b^\dagger(E,\mathbf{k}) b(E,\mathbf{k}) + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})}\, \mathbf{k}\, a^\dagger(\mathbf{k}) a(\mathbf{k}) \tag{10}$$

as well as the generators of three-dimensional rotations do not include the interaction:

$$J_{ij} = -i \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E}{(2\pi)^4 \kappa(E,\mathbf{k})} b^\dagger(E,\mathbf{k}) \times \left(k_j \frac{\partial}{\partial k_j} - k_j \frac{\partial}{\partial k_i}\right) b(E,\mathbf{k}) - i \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})} a^\dagger(\mathbf{k}) \times \left(k_j \frac{\partial}{\partial k_j} - k_j \frac{\partial}{\partial k_i}\right) a(\mathbf{k}). \tag{11}$$

Here indexes i, j mark coordinates k_i, k_j of 3-dimensional vector $\mathbf{k}$.

The interaction enters the Lorentz boost generators

$$\begin{aligned} J_{0i} = i \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} & \frac{\mathrm{d}E}{(2\pi)^4 \kappa(E,\mathbf{k})} b^\dagger(E,\mathbf{k}) \times \left(E \frac{\partial}{\partial k_j} + k_i \frac{\partial}{\partial E}\right) b(E,\mathbf{k}) \\ & + i \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})}\, a^\dagger(\mathbf{k}) \left(\omega(\mathbf{k}) \frac{\partial}{\partial k_j}\right) a(\mathbf{k}) \\ & + i \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E}{(2\pi)^3 2\omega(\mathbf{k})} \frac{\lambda\alpha(\kappa(E,\mathbf{k})}{\kappa(E,\mathbf{k})} \frac{1}{E} \\ & \times \left(E \frac{\partial}{\partial k_j} + k_i \frac{\partial}{\partial E}\right) \left[b(E,\mathbf{k}) + b^\dagger(E,-\mathbf{k})\right] \\ & \times \left[a(-\mathbf{k}) + \mathbf{a}^\dagger(\mathbf{k})\right]. \end{aligned} \tag{12}$$

We are interested in a particular case, $M > 2m$, when the system becomes Poincaré non-integrable [4]. In this case, the solution of the eigenvalue problem

$$\begin{aligned} [H, B^\dagger(E,\mathbf{k})] &= E\, B^\dagger(E,\mathbf{k}) \\ [H, B(E,\mathbf{k})] &= -E\, B(E,\mathbf{k}) \end{aligned} \tag{13}$$

(where $[\,\cdot\,,\,\cdot\,]$ is a commutator) is given by the following new creation and annihilation operators diagonalizing the Hamiltonian:

$$\begin{aligned} B^\dagger_{\substack{\mathrm{in}\\ \mathrm{out}}}(E,\mathbf{k}) &= b^\dagger(E,\mathbf{k}) + 2\pi\lambda\alpha(k(E,\mathbf{k})) G^\pm(E,\mathbf{k}) \\ \times \Bigg\{ & \int\limits_{E_0(\mathbf{k})}^{\infty} \mathrm{d}E' \frac{\lambda\alpha(k(E',\mathbf{k}))}{k(E',\mathbf{k})} \left(\frac{b^\dagger(E',\mathbf{k})}{E' - E \mp i0} - \frac{b(E',-\mathbf{k})}{E' + E}\right) \\ & - \frac{(E + \omega(\mathbf{k}))a^\dagger(\mathbf{k}) + (E - \omega(\mathbf{k}))a(-\mathbf{k})}{2\omega(\mathbf{k})} \Bigg\}, \end{aligned} \tag{14}$$

$$\begin{aligned} B_{\substack{\mathrm{in}\\ \mathrm{out}}}(E,\mathbf{k}) &= b(E,\mathbf{k}) + 2\pi\lambda\alpha(k(E,\mathbf{k})) G^\mp(E,\mathbf{k}) \\ & \times \Bigg\{ \int\limits_{E_0(\mathbf{k})}^{\infty} \mathrm{d}E' \frac{\lambda\alpha(k(E',\mathbf{k}))}{k(E',\mathbf{k})} \\ & \times \left(\frac{b(E',\mathbf{k})}{E' - E \pm i0} - \frac{b^\dagger(E',-\mathbf{k})}{E' + E}\right) \\ & - \frac{(E + \omega(\mathbf{k}))a(\mathbf{k}) + (E - \omega(\mathbf{k}))a^\dagger(-\mathbf{k})}{2\omega(\mathbf{k})} \Bigg\}. \end{aligned} \tag{15}$$

The incoming (in) and outgoing (out) solutions correspond to the boundary values of the Green's function $G^\pm(E,\mathbf{k}) = G(E \pm i0,\mathbf{k})$ where

$$G(z,\mathbf{k}) = \Bigg(\omega(\mathbf{k})^2 - z^2 - \int\limits_{E_0(\mathbf{k})}^{\infty} \mathrm{d}E^2 \frac{2\pi\lambda^2\alpha^2(k(E,\mathbf{k}))}{k(E,\mathbf{k})} \frac{1}{E^2 - z^2}\Bigg)^{-1}. \tag{16}$$

This function is analytic in the z complex plane except for the cut on the real line $(-\infty, -E_0(\mathbf{k})] \bigcup [-E_0(\mathbf{k}), +\infty)$ and satisfies the following dispersion relation

$$G(E, \mathbf{k}) = \int_{E_0(k)}^{\infty} \mathrm{d}E'^2 \frac{2\pi\lambda^2\alpha^2(k(E', \mathbf{k}))}{k(E', \mathbf{k})} \frac{|G(E', \mathbf{k})|^2}{E'^2 - E^2}. \quad (17)$$

New operators (14), (15) satisfy the commutation relations for bosons:

$$\left[B_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k}), B^\dagger_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k})\right] = (2\pi)^4 \kappa(E, \mathbf{k})\delta(E - E')\delta^3(\mathbf{k} - \mathbf{k}'). \quad (18)$$

The generators of the Poicaré group are diagonal in the new representation

$$P_\mu = \int \int_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}^3\mathbf{k}\, \mathrm{d}E}{(2\pi)^4\kappa(E, \mathbf{k})} \times k_\mu B^\dagger_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k}) B_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k}) \quad (19)$$

where $P_\mu = (P_0, \mathbf{P})$. Three dimensional rotations do not involve interaction terms (11) and therefore, present no difficulties. The commutators of the new operators with the Lorentz boosts are

$$\left[J_{0i}, B^\dagger_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k})\right] = i\left(E\frac{\partial}{\partial k_j} + k_i\frac{\partial}{\partial k_i}\right) B^\dagger_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k}). \quad (20)$$

In the original representation, the Fock space of states is spanned by the results of the action of the old creation operators on the bare vacuum state $|0\rangle$, which is defined as common zero eigenstate of the old annihilation operators

$$a(\mathbf{k})|0\rangle = b(E, \mathbf{k})|0\rangle = 0\,, \forall E, \mathbf{k}.$$

However, due to virtual transitions this is no more the vacuum state for the whole system and for the new operators we have $B_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k})|0\rangle \neq 0$. The new relativistic invariant vacuum state $|\Omega\rangle$ satisfying $B_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k})|\Omega\rangle = 0$ has the form

$$|\Omega\rangle = C_0 e^{V_{\text{vac}}}|0\rangle \quad (21)$$

where C_0 is the normalization factor and the transformation operator

$$\begin{aligned} V_{\text{vac}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2(\omega(\mathbf{k}) + \xi(\mathbf{k}))} \times \Bigg\{ \int_{E_0(\mathbf{k})}^{\infty} \mathrm{d}E \frac{\lambda\alpha(\kappa(E, \mathbf{k}))}{\kappa(E, \mathbf{k})} \gamma(E, \mathbf{k}) b^\dagger(E, \mathbf{k}) a^\dagger(-\mathbf{k}) \\ - \frac{\xi(\mathbf{k})}{2\omega(\mathbf{k})} a^\dagger(\mathbf{k}) a^\dagger(-\mathbf{k}) \\ + \int_{E_0(\mathbf{k})}^{\infty} \mathrm{d}E\mathrm{d}E' \frac{\lambda^2\alpha(\kappa(E, \mathbf{k}))\alpha(\kappa(E', \mathbf{k}))}{\kappa(E, \mathbf{k})\kappa(E', \mathbf{k})} \\ \times \gamma(E, \mathbf{k})\gamma(E', \mathbf{k}) \left(\frac{1}{2} + \frac{\omega(\mathbf{k}) + \xi(\mathbf{k})}{E + E'}\right) \\ \times b^\dagger(E, \mathbf{k}) b^\dagger(E, -\mathbf{k}) \Bigg\} \end{aligned} \quad (22)$$

is determined using the factorization of the Green's function [4]

$$G(E, \mathbf{k}) = \gamma(E, \mathbf{k})\gamma(-E, \mathbf{k}),$$

$$\gamma(E, \mathbf{k})^{-1} = -E - (\omega(\mathbf{k}) + 2\xi(\mathbf{k})) + O(1/E) \quad (23)$$

where function $\gamma(E, \mathbf{k})$ has a cut $(-\infty; -E_0(\mathbf{k})]$ [4] and $\xi(\mathbf{k})$ is a finite λ^2 correction to the term of the order of $O(1)$ in the variable E. It is a correction to the vacuum energy due to interactions.

The new creation operators $B^\dagger_{\substack{\text{in}\\\text{out}}}(E, \mathbf{k})$ acting on the new vacuum state $|\Omega\rangle$ define a new Fock space so that one particle state can be expressed in the form

$$\begin{aligned} |\Phi_{\text{in}}(E, \mathbf{k})\rangle = B^\dagger_{\text{in}}(E, \mathbf{k})|\Omega\rangle \\ = \Bigg\{ b^\dagger(E, \mathbf{k}) + \frac{\pi\lambda\alpha(\kappa(E, \mathbf{k}))}{\omega(\mathbf{k}) + \xi(\mathbf{k})} \gamma(-E - i0, \mathbf{k}) \\ \times \Bigg[\int_{E_0(\mathbf{k})}^{\infty} \mathrm{d}E' \frac{\lambda\alpha(\kappa(E', \mathbf{k}))}{\kappa(E', \mathbf{k})} \\ \times \gamma(E', \mathbf{k}) \left(1 + \frac{2(\omega(\mathbf{k}) + \xi(\mathbf{k}))}{E' - E - i0}\right) \\ \times b^\dagger(E', \mathbf{k}) + a^\dagger(\mathbf{k}) \Bigg] \Bigg\} |\Omega\rangle. \end{aligned} \quad (24)$$

Like in the non-relativistic Friedrich's model [3, 5, 7] the expression for the one particle eigenstate of the total Hamiltonian contains a partial resolvent. Its role in our model is played by $\gamma(-E - i0, \mathbf{k})$, which has a cut on the positive semi-axes of the energy plane $[E(\mathbf{k}), \infty)$ and shares its possible poles with the Green's function $G(E, \mathbf{k})$.

At this point we have completed a diagonalization of the Hamiltonian of two interacting fields. The Hamiltonian in this new representation is just a collection of non-interacting harmonic oscillators.

However, as it is shown in the next section, one can obtain another representation where some of the eigenstates obey irreversible damped dynamical laws.

3. Decaying Gamow modes

We remark first that in our case, $M > 2m$, the new representation contains no local field whereas, when $M < 2m$, there exist additional solutions $A^\dagger(\mathbf{k})$, $A(\mathbf{k})$ of (13) corresponding to a simple pole of $\gamma(-E, \mathbf{k})$ on the real line out of the cut. Together with the operators $B^\dagger_{\substack{\text{in}\\ \text{out}}}(E, \mathbf{k})$, $B_{\substack{\text{in}\\ \text{out}}}(E, \mathbf{k})$ these additional solutions form a complete set in the whole space which is related to the original complete set of $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$, $b^\dagger(E, \mathbf{k})$, and $b(E, \mathbf{k})$ through Bogolubov transformation. This transformation establishes one-to-one correspondence between the two complete sets. In our case, there is no pole of the Green's function $G(z, \mathbf{k})$ in the physical sheet of the complex energy plane (the first Riemann sheet in the variable z). Therefore, the only solutions of (13) are $B^\dagger_{\substack{\text{in}\\ \text{out}}}(E, \mathbf{k})$ and $B_{\substack{\text{in}\\ \text{out}}}(E, \mathbf{k})$ operators. Although, the Bogolubov transformation relating the old and new operators exists in this case as well, this transformation is no more unitary because there is no the one-to-one correspondence between the old and new operators. However, the Green's function $G(E, \mathbf{k})$ possesses poles in the second Riemann sheet of the complex plane. We denote as $\mu_c^2 = \mu^2 - i\mu\Gamma$, ($\mu$ and Γ are real) the pole of $G(E, \mathbf{k})$ as a function of the Lorentz invariant variable $E^2 - \mathbf{k}^2$, which is analytically continued through the cut from above in the right half plane. As we shall see below, μ_c may be considered as the complex mass of the Gamow state corresponding to this pole.

The corresponding poles $\pm z(\mathbf{k})$ of $G(E, \mathbf{k})$ as functions of E analytically continued through the cut from above in the right half plane and from below in the left half plane become functions of $\mathbf{k}$ such that $z(\mathbf{k}) = \left(\mathbf{k}^2 + \mu_c^2\right)^{1/2} = \tilde{\omega}(\mathbf{k}) - i\gamma(\mathbf{k})$ and $\gamma(\mathbf{k}) = \mu\Gamma/(2\tilde{\omega}(\mathbf{k}))$. Wwe notice that $\gamma(\mathbf{k})$ is not $\gamma(E, \mathbf{k})$). We see that the imaginary part of the pole, $\gamma(\mathbf{k})$, which is responsible for the life time of the field mode depends now on the momentum $\mathbf{k}$. For the weak coupling case, it reduces to the usually supposed expression $\gamma(\mathbf{k}) = M\Gamma/(2\omega(\mathbf{k}))$, which is valid only approximately up to the λ^4 terms. In the vicinity of the pole $z(\mathbf{k})$ the Green's function $G^+(E, \mathbf{k})$ has the following behaviour

$$G^+(E, \mathbf{k}) = \gamma^+(E, \mathbf{k})\gamma^+(-E, \mathbf{k}) = \frac{R(\mathbf{k})}{(\mu_c^2 - \mathbf{k}^2) - E^2} + \text{regular part} \quad (25)$$

where $R(\mathbf{k})$ is the residue of $G(E, \mathbf{k})$ in the pole. Therefore in the vicinity of the pole, one has

$$\gamma^+(\pm E, \mathbf{k}) = -\frac{\sqrt{R(\mathbf{k})}}{(\mu_c^2 + \mathbf{k}^2)^{1/2} \pm E} + \text{regular part}. \quad (26)$$

Using this fact we can rewrite (24) as

$$|\Phi_{\text{in}}(E, \mathbf{k})\rangle = \frac{1}{E - z(\mathbf{k})}|\Phi^{\text{G}}(\mathbf{k})\rangle + \text{regular part} \quad (27)$$

where $|\Phi^{\text{G}}(\mathbf{k})\rangle$ is the Gamow vector associated with the resonance pole $z(\mathbf{k})$. These vectors acquire meaning in the extended space where the Gamow kets are considered as distributions acting on the test functions represented by the bra vectors

$$\langle\psi| = \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E}{(2\pi)^4\kappa(E, \mathbf{k})} \times \psi(E, \mathbf{k})\langle\Omega|B_{\text{out}}(E, \mathbf{k}). \quad (28)$$

Then combining (24) and (28) we come to

$$\langle\psi|\Phi_{\text{in}}(E, \mathbf{k})\rangle = \psi(E, \mathbf{k})\frac{\gamma^+(-E, \mathbf{k})}{\gamma^-(-E, \mathbf{k})}. \quad (29)$$

Note that the bra vectors (28) are chosen such that the wave packets of the bra modes determined by the operator $B_{\text{out}}(E, \mathrm{k})$ including the function $G^+(E, \mathbf{k})$ can be analytically continued through the cut from above [11]. This is necessary for calculations of the pole contribution. As the result,the function $\gamma^+(-E, \mathbf{k})$ in (29) being analytically continued through the cut to the lower half plane has a pole at $z(\mathbf{k})$ and the analytic continuation of $\gamma^-(-E, \mathbf{k})$ to the lower half plane has neither singularity nor zero at this point. Restricting the variety of admissible functions $\psi(E, \mathbf{k})$ to a set of appropriate functions with good properties we see that $\langle\psi(E, \mathbf{k})|\Phi_{\text{in}}(E, \mathbf{k})\rangle$ admits a meromorphic extension to the lower half plane with a pole $z(\mathbf{k})$. Extracting the pole contribution from (29) we define

$$\psi(z(\mathbf{k}), \mathbf{k})\frac{\sqrt{R(\mathbf{k})}}{\gamma^-(-z(\mathbf{k}), \mathbf{k})} \equiv C(\mathbf{k})\langle\psi|\Phi^{\text{G}}(\mathbf{k})\rangle \quad (30)$$

where $|\Phi^{\text{G}}(\mathbf{k})\rangle$ is the Gamow state, $R(\mathbf{k})$ is the residue at the pole, and $\mathrm{C}(\mathbf{k})$ is the normalization constant. Using the fact that $|\Phi_{\text{in}}(E, \mathbf{k})\rangle$ is an eigenstate of the total Hamiltonian

$$P_0|\Phi_{\text{in}}(E, \mathbf{k})\rangle = E|\Phi_{\text{in}}(E, \mathbf{k})\rangle \quad (31)$$

we conclude that the Gamow state is also an eigenstate of the total Hamiltonian with complex eigenvalue [4, 11, 12]

$$P_0|\Phi^{\text{G}}(\mathbf{k})\rangle = z(\mathbf{k})|\Phi^{\text{G}}(\mathbf{k})\rangle. \quad (32)$$

Then the time evolution

$$e^{-itP_0}|\Phi^{\mathrm{G}}(\mathbf{k})\rangle \\ = e^{-iz(\mathbf{k})t}|\Phi^{\mathrm{G}}(\mathbf{k})\rangle = e^{-i\tilde{\omega}(\mathbf{k})t}e^{-\gamma(\mathbf{k})t}|\Phi^{\mathrm{G}}(\mathbf{k})\rangle. \quad (33)$$

shows the exponential decay of Gamow modes.

4. Lorentz transformation of decaying states

We describe here how the Lorentz boosts transform the exponentially decaying states. Our point of view is based on the fact that in scattering experiments, in the laboratory frame, one observes the wave packets with real momenta. Therefore the Lorentz transformation of the Gamow states should not change the reality of the momentum. As we shall see this will result in taking the point of view that in the irreversible description the Lorentz boosts act only on the wave packets without point-wise implementation.

We start with the transformation of the eigenstates of the Hamiltonian. Using the commutator of the new operators with the Lorentz boost and the Lorentz invariance of the vacuum state, we have for the eigenstate of the total Hamiltonian $|\Phi_{\mathrm{in}}(E,\mathbf{k})\rangle$ (24)

$$J_{o,i}|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle \\ = i\left(E\frac{\partial}{\partial k_j} + k_i\frac{\partial}{\partial k_i}\right)|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle\,. \quad (34)$$

Then the Lorentz transformation generated by (34) is

$$U(\alpha) = \exp\{i\alpha_i J_{0i}\}, \qquad \alpha_i = \alpha n_i\,, \\ \mathbf{n}^2 = 1\,, \qquad i = 1,2,3. \quad (35)$$

This transformation acts on $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$ as

$$U(\alpha)|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle = |\Psi_{\mathrm{in}}(E'_\alpha,\mathbf{k}'_\alpha)\rangle \quad (36)$$

where

$$E'_\alpha = E\cosh\alpha - (\mathbf{nk})\sinh\alpha, \\ \mathbf{k}'_\alpha = \mathbf{k} - \mathbf{n}((\mathbf{kn})\cosh\alpha - E\sinh\alpha). \quad (37)$$

We shall now extend the action of $U(\alpha)$ on the Gamow states. First, we try to do it applying $U(\alpha)$ point wise, i.e., for each Gamow mode separately. In order to do this, we isolate the exponentially decaying state by performing an analytic extension to the complex energy plane nearby the maximum on the real line where we expect to find a resonance. Extracting the pole contribution (in the complex E plane) from both parts of Eq. (36) we come to a formal transformation for the Gamow state

$$U(\alpha)|\Phi^{\mathrm{G}}(\mathbf{k})\rangle = |\Phi^{\mathrm{G}}(\mathbf{k}'_\alpha)\rangle, \\ \mathbf{k}'_\alpha = \mathbf{k} - \mathbf{n}((\mathbf{kn})\cosh\alpha - z(\mathbf{k})\sinh\alpha). \quad (38)$$

This leads to a complex momentum $\mathbf{k}'_\alpha$ of the transformed state. The complex value of the momentum is not admissible from physical point of view. In addition to this complex momentum another natural question to this scheme arises. In the reference frame in which we have obtained the Gamow state $|\Phi^{\mathrm{G}}(\mathbf{k})\rangle$ as a pole contribution to $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$, the momentum $\mathbf{k}$ is real. The same state from the point of view of any observer moving in this reference frame will have complex momentum. Why the reference frame in which we obtained the Gamow vector first is so exceptional? Therefore, we conclude that point-wise action of the Lorentz boost on the Gamow modes is not justified.

We suggest to overcome this difficulty by applying the Lorentz boost not to the Gamow state but to the original eigenstate of the full energy-momentum 4-vector, from which the Gamow state was obtained. Such transformation of the original state in the Hilbert space corresponds to the change of the reference frame form the initial one to the reference frame of the moving observer. In the new reference frame, the state $|\Psi(E,\mathbf{k})\rangle$ will have different energy and momentum due to the Lorentz boost. The energy and momentum of the transformed state $|\Psi(E',\mathbf{k}')\rangle$ are obtained according to Eq. (37). The Gamow state $|\Phi^{\mathrm{G}}(\mathbf{k})\rangle$ in the initial reference frame is obtained by analytic continuation in the variable E and the transformed Gamow state $|\Phi^{\mathrm{G}}(\mathbf{k}')\rangle$, i.e. the same state but in the moving reference frame, is obtained by analytic continuation of $|\Psi(E',\mathbf{k}')\rangle$ in the variable E' for the fixed value $\mathbf{k}'$.

This procedure, however, has an ambiguity, because when we continue analytically $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$ we do not fix E as it is the variable, as we use for the analytic continuation. But in order to fix $\mathbf{k}'$ after the Lorentz transformation of $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$, we have to know both E and $\mathbf{k}$ in the initial state, which is not the case as we do not fix E.

We avoid this apparent ambiguity using the Gamow wave packets, which we define as follows. We form first a one-particle wave packet from the energy-momentum eigenstates in the Hilbert space

$$|f\rangle = \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E}{(2\pi)^4\kappa(E,\mathbf{k})} \\ \times f(E,\mathbf{k})|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle \quad (39)$$

where the function $f(E,\mathbf{k})$ determines the shape of the wave packet. When we describe a particle with energy E_0 and the momentum $\mathbf{k}_0$ we consider a wave packet localized in the momentum representation around the values E_0 and $\mathbf{k}_0$. It is the function $f(E,\mathbf{k})$ that determines this localization giving appropriate “weight” to the field modes $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$ thus “shaping” the wave packet around E_0 and $\mathbf{k}_0$.

Extracting the pole contribution from the integrand of (39) we obtain a Gamow wave packet

$$|f^{\mathrm{G}}\rangle = \int \mathrm{d}^3\mathbf{k} \frac{f(z(\mathbf{k}),\mathbf{k})}{(2\pi)^4\kappa(z(\mathbf{k}),\mathbf{k})}|\Phi^{\mathrm{G}}(\mathbf{k})\rangle. \qquad (40)$$

The state of the Gamow wave packet is represented now by complex function $f(z(\mathbf{k}),\mathbf{k})$. However, it still shapes the wave packet around some real values of the momentum $\mathbf{k}^{\mathrm{G}}$ in the same way as $f(E,\mathbf{k})$ does.

After the extraction of the pole contribution we have a remaining part $|f^{\mathrm{bg}}\rangle = |f\rangle - |f^{\mathrm{G}}\rangle$, which is usually called "background" and can be written as

$$|f^{\mathrm{bg}}\rangle = \int \mathrm{d}^3\mathbf{k} \int\limits_{C_{z(\mathbf{k})}} \frac{\mathrm{d}E}{(2\pi)^4\kappa(E,\mathbf{k})} \times f(E,\mathbf{k})|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle \qquad (41)$$

where the contour of the integration runs along the real line from $E_0(\mathbf{k})$ above the cut as in (39) but then on the way it goes to the second Riemann sheet below the cut of $G(E,\mathbf{k})$, which is present in $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$. It makes a loop around the pole $z(\mathbf{k})$, returns to the first sheet, and continues to infinity along the real line above the cut. Although the background gives negligible contribution to physically relevant expectation values in the exponential era of the decay, it has to be taken into account in the discussion of the representations of the Poincaré group.

In order to obtain the Lorentz transformation of the Gamow wave packet we start, as we have suggested, with the transformation of the initial wave packet $|f\rangle$ in the Hilbert space. This transformation replaces E and $\mathbf{k}$ in $|\Psi_{\mathrm{in}}(E,\mathbf{k})\rangle$ in (39) by E'_α, $\mathbf{k}'_\alpha$, which are given by (37)

$$U(\alpha)|f\rangle = \int \mathrm{d}^3\mathbf{k} \int\limits_{E_0(\mathbf{k})}^{\infty} \frac{\mathrm{d}E}{(2\pi)^4\kappa(E,\mathbf{k})} \times f(E,\mathbf{k})|\Psi_{\mathrm{in}}(E'_\alpha,\mathbf{k}'_\alpha)\rangle. \qquad (42)$$

In order to see the shape of the transformed wave packet we change the variables in (42) by the transformation, which is inverse to (37), and obtain $E = E(E'_\alpha,\mathbf{k}'_\alpha)$, $\mathbf{k} = \mathbf{k}(E'_\alpha,\mathbf{k}'_\alpha)$. Then using the properties of the Lorentz invariant measure in (42) we come to

$$U(\alpha)|f\rangle = \int \mathrm{d}^3\mathbf{k}' \times \int\limits_{E_0(\mathbf{k}')}^{\infty} \mathrm{d}E' \frac{f(E(E',\mathbf{k}'),\mathbf{k}(E',\mathbf{k}'))}{(2\pi)^4\kappa(E',\mathbf{k}')} \times |\Psi_{\mathrm{in}}(E',\mathbf{k}')\rangle \qquad (43)$$

where we have dropped index α from the dummy variables E'_α and $\mathbf{k}'_\alpha$. The shape of the transformed wave packet is determined by the new function $f(E(E',\mathbf{k}'),\mathbf{k}(E',\mathbf{k}'))$. This function shapes the wave packet around the other values E'_0 and $\mathbf{k}'_0$. Then we extract the pole contribution from (43) and obtain

$$U(\alpha)|f^{\mathrm{G}}\rangle = \int \mathrm{d}^3\mathbf{k}' \frac{f(E(z(\mathbf{k}'),\mathbf{k}'),\mathbf{k}(z(\mathbf{k}'),\mathbf{k}')}{(2\pi)^4\kappa(z(\mathbf{k}'),\mathbf{k}')} \times |\Phi^{\mathrm{G}}(\mathbf{k}')\rangle, \qquad (44)$$

which is a transformed Gamow wave packet shaped around another real value $\mathbf{k}^{\mathrm{G}\prime}$. The remaining part gives the transformed background

$$U(\alpha)|f^{\mathrm{bg}}\rangle = U(\alpha)|f\rangle - U(\alpha)|f^{\mathrm{G}}\rangle = \int \mathrm{d}^3\mathbf{k}' \int\limits_{C_z(\mathbf{k}')} \mathrm{d}E' \frac{f(E(E',\mathbf{k}'),\mathbf{k}(E',\mathbf{k}'))}{(2\pi)^4\kappa(E',\mathbf{k}')} \times |\Psi_{\mathrm{in}}(E',\mathbf{k}')\rangle. \qquad (45)$$

In this way we perform first transformation $U(\alpha)$ (35)-(37) of a wave packet from the laboratory to the moving frame. In the two frames, the same wave packet has different shapes (39) and (42), the last one being transformed by the appropriate Lotentz boost. Then by analytic extension in the complex energy plane we find the contribution of the Gamow modes into the wave packet in both frames (40) and (44). The momentum of both these packets is real. It is the transformation from the Gamow modes giving major contribution to the wave packet in the laboratory frame "at rest" to the Gamow modes contributing to the wave packet in the "moving frame" that we identify as the Lorentz transformation of exponentially decaying states.

5. Conclusion

We have constructed a new non-point-wise representation of the Poincaré group. This representation includes exponentially decaying Gamow states. The obtained transformation of the complex Gamow wave packet leads to a new wave packet which is again a combination of the Gamow states with real momentum. Thus, we have shown that relativistic transformation of the Gamow modes, which obey strictly exponential decay law, can be introduced in a consistent way with the help of decaying wave packets. This transformation keeps the momentum of Gamow modes real. We have seen that due to the physical requirement of the reality of the momentum $(z(\mathbf{k}),\mathbf{k})$ cannot be a 4-vector. Therefore the Lorentz boost in non-pointwise implementable.

The existence of Gamow states manifests intrinsic irreversibility of dynamical systems [3]. Therefore, by construction of the consistent Lorentz transformation for Gamow modes as a part of new representation of the Poincaré group, we have shown compatibility of irreversible evolution with relativistic theory. Another representation of the Poincaré group including Gamow states can be found in [13, 14].

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