

Algebrodynamics in Complex Space-Time and The Complex-Quaternionic Origin of Minkowski Geometry.

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Abstract

We present a scheme of biquaternionic algebrodynamics based on a nonlinear generalization of the Cauchy-Riemann “holomorphy” conditions considered therein as fundamental field equations. The automorphism group $SO(3, \mathbb{C})$ of the biquaternion algebra acts as a proper Lorentz group on a real space whose coordinates are bilinear in the complex coordinates of biquaternionic vector space. A new invariant of Lorentz transformations then arises — the geometric phase. This invariant can be responsible for the quantum properties of particles associated in this approach with field singularities. Some new notions are introduced, related to “hidden” complex dynamics: “observable” space-time, the ensemble of identical correlated particles-singularities (“duplicons”) and others.

1. Introduction. Geometric structures and fields in complex space-time

The complexification \mathbb{CM} of Minkowski space-time \mathbb{M} arises permanently in the framework of general relativity and field theories. The generally accepted complex nature of physical fields also implies the most natural realization in complex coordinate space. The holomorphic structure of field and coordinate spaces can essentially simplify the calculations of string diagrams and, after reduction to a Euclidean sector, ensures convergence of functional integrals.

It is well known that the “virtual” dynamics in \mathbb{CM} gives rise to a peculiar real dynamics in its Minkowski “cut”. For example, null congruences with *twist*, the Kerr congruence among them, may be viewed as generated by a point “charge” located at some point of \mathbb{CM} , and the Kerr parameter a has the meaning of charge separation from the real slice \mathbb{M} .

More generally, E.T. Newman et al. [1–3] have

demonstrated that the *complex null cone* of a point charge moving along an arbitrary world line in \mathbb{CM} forms a *shear-free null congruence* of rays which is related to asymptotically flat solutions of Maxwell, Einstein or Einstein-Maxwell equations with interesting particle-like properties, in particular, with the gyromagnetic ratio equal to that of a Dirac fermion (recall that Carter [4] was the first to notice this property of Kerr’s singular ring).

Note that it is just the *shear-free* null congruences that have defining equations *holomorphic* with respect to the principal null 2-spinor field λ_A of the congruence (i.e. do not contain the complex-conjugated spinor $\lambda_{A'}$):

$$\lambda^A \lambda^B \nabla_{AA'} \lambda_B = 0, \quad A, B, \dots = 0, 1, \quad (1)$$

and can therefore be naturally extended analytically to the whole \mathbb{CM} space. A set of interesting physical fields, in particular, the Maxwell one, has been associated with such congruences by I. Robinson [5], P. Tod [6], E.T. Newman [7] and others. The electromagnetic field (as well as the complex Yang-Mills

field), with nontrivial gauge symmetries and with a *self-quantized electric charge* of its bounded singularities — sources, has been introduced in our works [8–12] in the general framework of the so-called *algebrodynamical* paradigm.

In *algebrodynamics* (see, e.g., [8, 9, 13–15] and references therein) one seeks a *numerical*, abstract origin of the physical laws and space-time geometry, i.e., a sort of *the Code of Nature*, reproducing, at the most elementary level, the *genetic* code. In particular, the exceptional algebra of *quaternions* \mathbb{Q} can be accepted as the fundamental *space-time algebra*. Since its invention by Sir William Hamilton in 1843, it is known that \mathbb{Q} possesses a perfect structure for describing our 3D space geometry since its *automorphism* group is (2:1) the group of 3-rotations $SO(3)$.

Moreover, Hamilton was in fact the first, nearly 50 years before Minkowski, to introduce the concept of unique *space-time* via identifying the fourth “real” unit of \mathbb{Q} -algebra with physical time. It was quite natural in the times of Hamilton since this unit remains invariant under the $SO(3)$ transformations as it should be for Newtonian absolute time. However, the relative nature of physical time and the fundamental Lorentz invariance established later in special relativity (SR) have come to an evident contradiction with the structure of the \mathbb{Q} algebra.

To escape this difficulty, the authors usually deal with a complex extension of \mathbb{Q} , the algebra of *biquaternions* \mathbb{B} . Then, on a coordinate subspace of \mathbb{C}^4 vector space of \mathbb{B} , one reveals the metric of real Minkowski space \mathbb{M} , and the theory on this slice is completely Lorentz-invariant.

In our previous works, we also used this scheme and thus constructed a Lorentz-invariant algebraic field theory. The fundamental physical field is therein represented by “hyper-holomorphic” *biquaternionic*-valued functions which satisfy the primary field equations, *the generalized Cauchy-Riemann conditions* proposed in 1980 [16] (see also [13, 17]). Remarkably, the latter are *nonlinear* owing to the property of *noncommutativity* of \mathbb{B} and are thus capable of describing (self-) interacting fields. For details of \mathbb{B} -algebrodynamics we refer the reader to our papers [8, 9, 11, 18, 19].

However, the restriction of four complex coordinates of \mathbb{B} to a real Minkowski subspace (which does not even form a subalgebra in \mathbb{B}) seems rather artificial. Moreover, the problem is even more difficult: we do not know any algebra whose automorphism group would be isomorphic to the Lorentz group of SR (or naturally contain the latter as its subgroup). It is clear that without solving this problem one would not be able to propose a consistent realization of the algebraic (*algebrodynamical*) programme.

Meanwhile, there is a peculiar property of \mathbb{B} algebra

that distinguishes it as a probable algebra of physical space-time: *its automorphism group* $SO(3, \mathbb{C})$ *is 6-parametric and closely related to the* $SL(2, \mathbb{C})$ *and thus to the (proper) Lorentz group*. However, it acts on the 3D complex (6D real) space which, at first glance, has no relation to 4D Minkowski space. Nonetheless, the \mathbb{B} algebra with its $SO(3, \mathbb{C})$ symmetry group have been used in a number of works, e.g., in the *quaternionic theory of relativity* developed by A.P. Yefremov [20, 21]. In this approach, one deals with three spatial and three temporal coordinates represented by the real and imaginary parts of (the linear space of) the complex vector units I, J, K of \mathbb{B} , respectively, whereas the fourth scalar unit is “frozen” and is not involved in the physical dynamics. To reduce the 3D time to physical one-dimensional time, special orthogonality conditions are imposed.

Alternatively, in our recent papers [26, 27], the geometry of a *complex null cone* (CNC) in the \mathbb{C}^4 vector space of \mathbb{B} has been exploited. Point particles separated by a null complex interval have equal *twistor fields* and, as a consequence, *their dynamics is correlated*. In turn, the 6D “observable” space of the CNC naturally decomposes into 4D physical space-time and an orthogonal space of a 2-sphere (internal *spin* space). A physical time interval is defined by this as the *whole distance passed by a point particle in the complex space* $\{Z_\mu\} \in \mathbb{C}$ *with respect to the Euclidean metric* $\Sigma|Z_\mu|^2$.

However, all these and other perspective schemes (among which, an interesting treatment of (3+3) geometry by I.A. Urusowski [22, 23] should be certainly mentioned) suffer from an evident lack of coordination with SR since the role of Lorentz symmetries is vague. In this paper (section 2), we construct a nonlinear mapping of the \mathbb{C}^4 vector space of \mathbb{B} algebra into the *timelike* (including its null cone boundary) subspace of real Minkowski space on which the \mathbb{B} -automorphism group $SO(3, \mathbb{C})$ acts exactly as the Lorentz group. In other words, we show that physical (3+1) space-time, together with its internal causal structure, is in fact encoded in the internal properties of the biquaternion algebra which could thus be really treated as the space-time algebra.

In section 3 we briefly consider algebrodynamics in complex quaternionic space and its “projection” onto Minkowski space-time based on the above constructed mapping. The existence of a “hidden” complex space leads to a number of new concepts related to the dynamics of particles-singularities and, probably, to their quantum properties. In particular, we briefly discuss the structure of “observable” space-time, the procedure of formation of an ensemble of identical correlated “duplicons” and the “evolutionary” meaning of the proper complex time coordinate.

2. Biquaternion algebra as the origin of Minkowski space-time

The complex quaternion (biquaternion) algebra \mathbb{B} has an exact representation as the full 2×2 complex matrix algebra, i.e., the following one ($\forall Z \in \mathbb{B}$):

$$Z = \begin{pmatrix} u & w \\ p & v \end{pmatrix} = \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}. \quad (2)$$

Here $\{u, w, p, v\} \in \mathbb{C}$ or $\{z_0, z_1, z_2, z_3\} \in \mathbb{C}$ are four complex coordinates, matrix or Cartesian, respectively.

Under the 3C-parameter transformations of the (internal) automorphism group $SO(3, \mathbb{C})$, preserving multiplication in \mathbb{B} , one has

$$Z \mapsto \bar{Z} = M * Z * M^{-1}, \quad (3)$$

$\forall M \in \mathbb{B} : \det M = 1$. Three “vector” coordinates $\{z_a\}$, $a = 1, 2, 3$ undergo 3D complex rotations whereas the “scalar” coordinate z_0 corresponding to the trace part of the Z -matrix remains invariant. Another invariant of these transformations is the determinant of the Z -matrix, the complex (indefinite) metric in \mathbb{C}^4 ,

$$\gamma = \det Z = (z_0)^2 - (z_1)^2 - (z_2)^2 - (z_3)^2 = \text{inv}. \quad (4)$$

Let us here stress that, despite the representation used above, this complex metric has no explicit relation to the Minkowski metric.

Due to invariance of the trace coordinate z_0 , one can take as the second independent invariant the 3D complex metric

$$\sigma = (z_1)^2 + (z_2)^2 + (z_3)^2, \quad (5)$$

which represents *two real-valued invariants*. The main of them is the (square of the) absolute value $S \in \mathbb{R}_+$:

$$S^2 = \sigma\sigma^* = (\text{Re } \sigma)^2 + (\Im \sigma)^2 = (|\mathbf{p}|^2 - |\mathbf{q}|^2)^2 + (2|\mathbf{p} \cdot \mathbf{q}|)^2, \quad (6)$$

where the two 3-vectors \mathbf{p} and \mathbf{q} represent the real and imaginary parts, respectively, of a complex coordinate 3-vector

$$z_a = p_a + iq_a, \quad a = 1, 2, 3; \quad (7)$$

the notations $|\dots|$, \cdot and \times (see below) stand for the usual operations of 3-vector algebra: the modulus, scalar and vector products, respectively.

By simple transformations using vector algebra, the expression (6) can be brought to the following equivalent form:

$$S^2 = (|\mathbf{p}|^2 + |\mathbf{q}|^2)^2 - (2|\mathbf{p} \times \mathbf{q}|)^2 \equiv T^2 - |\mathbf{X}|^2, \quad (8)$$

where the positive-definite coordinate

$$T = |\mathbf{p}|^2 + |\mathbf{q}|^2 \quad (9)$$

can be thought of as representing *physical time*, while the three real coordinates

$$\mathbf{X}_a = 2\varepsilon_{abc}p_bq_c \quad (10)$$

constitute the radius vector $\mathbf{X} = \{X_a\}$ of the Euclidean *physical 3-space*.

Under 3-rotations (3) by a *real-valued* angle, the coordinate T is invariant while the coordinates \mathbf{X} transform as an ordinary 3-vector. Therefore, we can always use these rotations to fix the plane formed by a pair of vectors $\{\mathbf{p}, \mathbf{q}\}$ so that one has, say, $z_3 = 0$, $X_1 = X_2 = 0$, whereas the real motion takes place in the direction X_3 orthogonal to the plane. One can then easily check that rotations in the $\{z_1, z_2\}$ plane by an *imaginary* angle $i\psi$ give rise to the corresponding *Lorentz boosts* for the quantities $\{T, X_3\}$, yet by a *double (hyperbolic) angle* 2ψ . Thus, Lorentz symmetry here manifests itself in a rather nontrivial way.

The main invariant (8) of the $SO(3, \mathbb{C})$ transformations can be thus identified with the metric interval of Minkowski space. However, being induced by a mapping from a 3D complex biquaternionic subspace, it is always positive-definite, i.e. corresponds to timelike or null separations of the corresponding points. Thus the existence of a primary complex space gives rise not only to the real (1+3) pseudo-Euclidean geometry, but also to the latter’s *causal structure* which, in SR theory, has to be postulated additionally.

If one now applies the above-constructed mapping to *increments* of the corresponding coordinates, then, for an infinitesimal spacial displacement $\delta\mathbf{X}$, one obtains

$$\delta\mathbf{X} = 2\delta\mathbf{p} \times \delta\mathbf{q}, \quad (11)$$

whereas for corresponding time interval δT one gets the positive-definite expression

$$\delta T = |\delta\mathbf{p}|^2 + |\delta\mathbf{q}|^2, \quad (12)$$

so that the resulting time appears to be *irreversible*. On the other hand, the expression (12) does not form a full differential (i.e., one here deals with an effective *non-holonomy* of the resulting space-time). As a result, any *closed* path $\mathbf{z} = \mathbf{z}(\lambda) = \{\mathbf{p}(\lambda), \mathbf{q}(\lambda)\}$, $\lambda \in \mathbb{R}$ of a point particle in the primary 3D complex space corresponds to a nonzero positive value of the physical time passed

$$\Delta T = \oint (|\mathbf{p}'(\lambda)|^2 + |\mathbf{q}'(\lambda)|^2)d\lambda. \quad (13)$$

This effect could be closely related to that of time delay in SR. As to the analogous effect of *spatial* non-holonomy, again due to bilinearity of the corresponding expression (11), it has no analogues in SR and deserves a special consideration. Generally, however, we expect that all the familiar effects specific of SR and based completely on the Lorentz invariance property of the Minkowski interval will be preserved in the suggested algebro-geometric scheme.

Let us now discuss the problem of restoration of the “hidden” complex space structure from that of Minkowski space-time (11), (12). It is noteworthy that the 3D complex null cone subspace

$$\delta\sigma = (\delta z_1)^2 + (\delta z_2)^2 + (\delta z_3)^2 = 0 \quad (14)$$

is mapped into the real light cone of Minkowski space-time

$$\delta S^2 = (\delta T)^2 - |\delta\mathbf{X}|^2 = 0. \quad (15)$$

Taking in account that in this case Eq.(14) leads to the constraints

$$|\delta\mathbf{p}| = |\delta\mathbf{q}|, \quad \delta\mathbf{p} \cdot \delta\mathbf{q} = 0, \quad (16)$$

one finds that, in order to restore the complex structure (i.e. the vectors $\delta\mathbf{p}, \delta\mathbf{q}$) from the real displacement $\delta\mathbf{X}$ (in this case $\delta T = |\delta\mathbf{X}|$), it is necessary to set the value of one additional angle (the *phase*). Indeed, the direction of $\delta\mathbf{X}$ fixes the orthogonal plane containing both vectors which, as a result of (16), are equal in modulus and mutually orthogonal. Thus they are fixed by $\delta\mathbf{X}$ up to a common 1-parameter rotation in this plane.

To pass to the general case, let us recall that the main invariant σ given by Eq. (5) is complex-valued, and, apart from the Minkowski interval, there exists another quantity, the phase α defined by the relation

$$\tan \alpha = \frac{2\delta\mathbf{p} \cdot \delta\mathbf{q}}{|\delta\mathbf{p}|^2 - |\delta\mathbf{q}|^2}; \quad (17)$$

it also remains invariant under the whole 6-parameter group of Lorentz transformations. For this reason, the “hidden” geometric phase α accompanying any displacement in Minkowski space-time is of fundamental importance and may probably have a direct relationship to the quantum interference phenomena. Note that, for a lightlike motion represented by (14), (15), the phase α is indefinite since both $\text{Re}(\sigma)$ and $\text{Im}(\sigma)$ turn to zero.

Let us now return to the problem of restoring the primary complex geometry in the case when, apart from δT and $\delta\mathbf{X}$, the value of the phase α is given. Without going into details of calculations, we only announce the answer: the vectors $\delta\mathbf{p}$ and $\delta\mathbf{q}$ of a complex displacement are fixed by these data, again up to a common rotation of this pair in the plane orthogonal to the $\delta\mathbf{X}$ direction. In particular, for the angle θ between these vectors one gets

$$\cos^2 \theta = \frac{1 - v^2}{1 + v^2 \cot^2 \alpha}, \quad (18)$$

$v \leq 1$ being the 3D velocity of motion in the units of fundamental light velocity $c = 1$. For a point at rest $v = 0$, in particular, the two vectors \mathbf{p}, \mathbf{q} are either parallel ($\theta = 0$) or antiparallel ($\theta = \pi$), and this duality might be related to the quantum spin properties of particles.

3. “Hidden” algebraic dynamics in the complex-quaternionic space and its image in Minkowski space-time

In biquaternionic (\mathbb{B}) algebrodynamics, any matrix component of the fundamental \mathbb{B} -field, due to the generalized Cauchy-Riemann conditions, satisfies the *complex eikonal equation* (CEE) [8, 9]. On the other hand, any solution of the CEE gives rise to null congruences of “rays”, one of which is *shear-free* [19]. An interesting class of null shear-free congruences (SFC) is generated by a “virtual” point-like charge “moving” along an arbitrary world line $Z = Z(\kappa)$, $\kappa \in \mathbb{C}$ in 4D complex space [24, 25].

In this case, a fundamental SFC is formed by a bundle of null rectilinear complex “rays” starting/ending on the charge. The projective twistor field of the congruence $\{\xi, \tau\}$ at a point $Z \in \mathbb{CM}$ is determined by the condition

$$(Z - \widehat{Z}(\kappa))\xi = 0, \quad \iff \quad \tau = Z\xi = \widehat{Z}(\kappa)\xi. \quad (19)$$

The field preserves its value along any of the rays, depends on the radial direction in the vicinity of a charge and is indefinite at the very point of its location. Thus the world line of the charge is just the *focal line* of the SFC.

The position of a charge which “affects” the point Z is fixed, on account of (19), by the *complex null cone* (CNC) equation

$$\det |Z - \widehat{Z}(\kappa)| = 0, \quad (20)$$

which determines the value of the corresponding parameter κ . Note that the field $\kappa(Z)$ identically satisfies the CEE.

However, contrary to the case of real Minkowski space-time \mathbf{M} , in the complex space \mathbb{CM} the CNC equation (20) has, generically, a *great number of roots* $\{\kappa_n\}$ and determines an *ensemble of identical point “particles”* \widehat{Z}_n which “affect” Z by the corresponding twistor field. In our paper [27] these were called *duplicons*.

In particular, one can take a point belonging to the world line itself, i.e., $Z = \widehat{Z}(\pi)$, and consider it as an *elementary observer*. From the CNC equation (20) one then obtains a set of parameters $\kappa = \kappa_n(\pi)$ and, accordingly, an ensemble of duplicons $\widehat{Z}_n(\pi)$. Note that in a real \mathbf{M} this construction is impossible since one has there the only solution of the “light cone” equation $\kappa = \pi$ (i.e. the “retardation time” turns to zero).

Thus, at any point of its world line $\widehat{Z}(\pi)$, an elementary observer deals essentially with a (6D real) subspace of a local CNC — with *observable* space-time [27]. In the *relative* coordinates $\bar{Z}(\pi) = \widehat{Z}(\pi) - \widehat{Z}(\kappa(\pi))$, the local CNC equation reads

$$(\bar{z}_1)^2 + (\bar{z}_2)^2 + (\bar{z}_3)^2 = (\bar{z}_0)^2 \quad (21)$$

and determines the observable space *at a fixed instant of the (complex) proper time of the observer* $z_0(\kappa)$. Note that the whole “matter-like” distribution of duplicons at this \bar{z}_0 with respect to the observer belongs to its local CNC space and, as we shall see now, corresponds to all different intervals of coordinate time \bar{T} *at its past*.

Indeed, let us now reduce the CNC (21) to real physical space-time making use of the mapping constructed in section 2. Taking the modulus of both sides of (21), we get on account of (5), (6) and (8):

$$\bar{T}^2 - |\bar{\mathbf{X}}|^2 = S^2 \equiv |\bar{z}_0|^2. \quad (22)$$

Thus we see that *the real Minkowski interval is nothing but the modulus of the (square of) complex proper time of an elementary observer*. As to the other, *phase invariant* (17), it obviously corresponds to the phase of the (square of) complex proper time \bar{z}_0 .

It has been shown in [26,27] that the *absolute* complex coordinate z_0 can be considered globally as the *evolutional parameter* along rectilinear complex rays of the null SFC which preserves both the value of the twistor field of the congruence and *the caustic structure*. In other words, the caustics propagate along *some* of the complex rays connecting the observer with its “images”, duplicons. On the other hand, the coordinate z_0 parametrizes the location of an elementary observer on its world line and is responsible for all alterations of the “picture of the World” perceived by the observer through reception of signals-caustics from the duplicons and through permanent observation of temporal dynamics of the latters correlated by the primary twistor field.

One of the peculiar features of complex time dynamics is the absence of the *event ordering* which, however, can be restored by introduction of an extra structure, the *evolutionary curve* [26,27]. Along with variations of the α -phase invariant, this can open a way to explanation of the quantum uncertainty and the interference phenomena.

4. Was Hamilton’s dream on the quaternionic structure of the Universe indeed true?

To conclude, in this paper we have proposed to consider the observable Minkowski space-time as a “shadow space” of a “hidden” fundamental geometry which has a purely abstract, “numerical” origin and is fully encoded in the structure of the biquaternion algebra. Just in the spirit of Hamilton, the fourth (scalar) coordinate in \mathbb{B} is closely related to time, but this is *proper time*, and this time is *complex-valued*. On the other hand, Minkowski space itself, *together with its internal causal structure*, is encoded in the main 3D complex subspace of \mathbb{B} and is *bilinearly* determined by the

complex coordinates of the latter, through the structure of the \mathbb{B} -automorphism group $SO(3, \mathbb{C})$. Moreover, another *phase invariant* of the Lorentz transformations arises naturally in the approach. It seems that we have met with quite a new “hidden” geometry of nature which by itself predetermines the kinematics and interactions of particle-like formations, their spin properties and quantum uncertainty in general. We hope that these expectations are really well grounded.

Note added in pursuit. Concept of the complex space-time has been proposed repeatedly in the frameworks of quantum mechanics (see, e.g., [28,29]), field theory [30,31] and General Relativity (see numerous papers of R. Penrose, E.T. Newman, A. Trautman et al.). In an algebraic approach which makes use of complex quaternions the adoption of this concept seems inevitable. However, up to now the *geometrophysical* sense of the additional “imaginary” dimensions, as well as the exceptional position of the Minkowski space-time in the structure of complexified space have not been understood at all. We hope that this very article will give impetus to reconsideration and intensive development of various approaches dealing with complexification of space-time, of the algebrodynamics for the first turn.

Acknowledgements

The Editorial Board of EMPH kindly agreed to make here a reprint of this article published firstly in the Journal of Russian Gravitational Society: GRAVITATION & COSMOLOGY, Vol.11, December 2005, pages 354-358. The author is also grateful to the editors of G&C for permission to reprint the paper.

Manuscript received May 12, 2006

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