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Local Hamiltonization and Foliation: A New Solution to the Hamiltonization Problem

Contents

Abstract

The transformation of a set of arbitrary ordinary differential equations (odes) in n-dimensions to a Hamiltonian form is known as the Hamiltonization problem. A solution to this problem is found in the Lie-Königs theorem which aims to solve the set of odes. In 1992, Volker Perlick (see ref. [3]), generalized the theorem using using invariant differential geometrical terminology, but his results required, as do those of Lie and Königs, the complete solution to the set of odes. It was in 1996 (see ref. [12]) when Sergio Hojman essentially changed the situation of the Hamiltonization problem by proposing a solution that does not require a complete solution to the set of odes, thereby relaxing the conditions of the Lie-Königs theorem. In this paper, we generalize the Hojman framework to solve the local Hamiltonization problem without a full explicit solution for the set of odes, and we construct a new Poisson tensor for the harmonic oscillator which was not obtained by Hojman.

1. Introduction

The Hamiltonization problem is the question of whether or not an arbitrary set of ordinary differential equations (odes) admits a Hamiltonian formulation. In 1996 Sergio Hojman was able to succesfully construct a Hamiltonian formulation [11] for a set of odes in n-dimensions defined by an arbitrary smooth vector field X. This construction required an additional vector field η and a first integral f of X related by the following equations (written in an invariant notation not used by Hojman),

$$\begin{aligned} [\eta, X] &= 0, \\ \iota_X df &= 0 \end{aligned} \qquad (A)$$

or

$$\begin{aligned} [\eta, X] &= aX, \\ \iota_X df &= 0 \end{aligned} \qquad (B)$$

where $[,]$ is the Lie bracket, ι_X the contraction operator and d the exterior derivative. The unknowns in these equations are η and H with X a given data. If we have η and H the Poisson tensor constructed by Hojman is given by $P = -\frac{\eta \wedge X}{\iota_{df}\eta}$ and our odes $\gamma_* \frac{d}{dt} = \gamma_* X$, for any smooth curve γ, can be written in the non-canonical Hamiltonian form,

$$\begin{aligned} \gamma_* \frac{d}{dt} &= \gamma_* \iota_{df} P = -\gamma_* \iota_{df} \left(\frac{\eta \wedge X}{\iota_{df}\eta} \right) \\ &= -\gamma_* \left(\frac{(\iota_{df} X)\eta - (\iota_{df}\eta)X}{\iota_{df}\eta} \right) = \gamma_* X \end{aligned}$$

as required.

The Hojman construction is a novel solution to an old problem in mathematical-physics known as the inverse problem of the calculus of variations. To obtain the Hamiltonian formulation, it was necessary to use a regular Lagrangian formulation for the odes (see [35–39]). This was the first answer to the Hamiltonization question and can be found in any standard textbook on classical mechanics (for example [39]). Subsequently S. Lie [19] and G. Königs [28] were able to avoid the use of a Lagrangian formulation by introducing integral invariants and Pffaf´s reduction method of non-exact 1-forms (see [2,18] and [22]) in order to address the question of Hamiltonization. This second answer, known as the Lie-Königs theorem (see section II, below) requires a complete solution to the set of odes in the form of n-integrals and can be found in E. T. Whittaker´s classical treatise [1]. Finally, Hojman developed what would become a third answer to the question which required one integral of motion f and a vector field η related by the equations (A) or (B). Hence, in the later work of Hojman the conditions of the Lie-Königs theorem are relaxed, because weaker hypothesis are required.

The Lie-Königs theorem and its applications to mechanics were more or less ignored (see [3] and [34]) until 1992 when Volker Perlick, using a global geometric formulation, was able to show that a dynamical system admits a Hamiltonian formulation if the underlying manifold admits a canonical symplectic 2-form (see proposition 2 in [3]).

Although he used a global geometric formulation, Perlick did not relax the conditions of the theorem because he used a complete solution to the set of odes to obtain the Hamiltonian formulation. He had a very powerful reason to doing this, because many important theorems (the Darboux [5], the Liouville [10], the flow box [33], and the symplectic stratification theorem [5] theorems among others) are proved using the supposition that we can construct the complete solution to the odes in question. Nevertheless, Perlick´s approach represents progress regarding the Hamiltonization problem because of his modern differential geometric techniques and clarification of some underlying hypothesis.

Following Perlick, there have been several interesting attempts to avoid the use of a complete solution to the odes, such as in the work of Gümral and Nutku [29] and Goedert, Haas, Hua, Feix and Cairo [30]. Unfortunately, the manifold dimension is restricted to 3 and the results are not general. However, the results and geometrical framework of Gümral and Nutku are closely related to our approach, with the exception of our use of the dual representation (see section VII of [29]).

The work of G. R. W. Quispel and H. W. Capel [31], which treated the Hamiltonization problem in n-dimensions (see [32] for their results using a nonglobal differential geometric formalism), appears around the same time as Hojman´s work. More recently, Torres del Castillo and Mendoza Torres [6], have developed a treatment of the problem which, although neglecting the Darboux reduction hypothesis, requires non-singular Poisson or symplectic tensors.

In this paper, our main objective is to generalize the results of S. Hojman for finite dimensions using modern differential geometric techniques in order to solve the Hamiltonization problem with new and weaker assumptions than Lie, Königs and Perlick. These new and weaker conditions are essentially contained, but nor fully developed, in Hojman´s work cited here. The generalization that we obtain can be used to prove theoretical results such as a converse to the symplectic stratification theorem (see [5] p. 302).

This paper is organized as follows: in section II we explain the Lie-Königs theorem in detail following the Whittaker and the Perlick approaches, in order to obtain the set of underlying theoretical hypotheses. We then explain Hojman´s theoretical hypotheses and introduce our theoretical approach to Hamiltonian systems. In section III, we provide all necessary definitions and provide a distinction of Poisson tensors using rank conditions. In section IV, we begin to treat the almost-Poisson case, establishing the most elementary results. We present the pre-Poisson case and equations (42a–d) in section V which represent the generalization of Hojman´s theory.

We begin with a discussion of two important conditions in the theory of Poisson manifolds indicating that they are in fact independent conditions. Although not usually found in the literature, we require this indication to motivate some of the results. We also introduce the notion of "pre-Poisson" tensor again using a distinction based on rank conditions. Finally, in section VI, we establish the Hamiltonization problem and its solution using equations (42a–d). We offer an example using the harmonic oscillator through which we obtain a Poisson tensor not previously obtained by Hojman in [11]. Our conclusions are presented in Section VII.

2. The Nature of the Hamiltonization Problem

In this section the theorems of Lie-Königs and Hojman are introduced, explained and compared.

According to Whittaker (see [1] art. 116,117 and 118), the strongest and by far most difficult condition to satisfy when treating the Hamiltonization problem is that of solving the order n set of odes in order to obtain a solution in an implicit form such as:

$$\xi_1(x_1,...,x_n,t)=c_1,...,\xi_n(x_1,...,x_n,t)=c_n, \qquad (1)$$

where $\{\xi_1,..,\xi_n\}$ is a functionally independent set of algebraic first integrals of motion, suitable for writing down integral invariants and applying Pffaf ´s reduction procedure. (See [18] chapter IV and [22] chapter VI. For up to date information see [2] p. 77–112).

If we have the set of integrals (1) for any arbitrary set of time independent functions N_i, the associated integral invariant is $I = \int \sum_{i=1}^{n} N_i(\xi_1,...,\xi_n)d\xi_i$, which can be transformed into $\int \sum_{j=1}^{n+1} M_j(x_1,...,x_n,x_{n+1})dx_j$ with $t = x_{n+1}$and $M_j = \sum_i N_i(\xi_1(x_1,...,x_n,t),...,\xi_n(x_1,...,x_n,t))\frac{\partial \xi_i}{\partial x_j}$. Using this integral invariant as a starting point, Whittaker applies Pffaf´s results on the reduction of non-exact 1-forms to the integrand of the integral invariant I to get a canonical 1-form $\sum_{i=1}^{k} p_i(x)dq_i(x) - dS(x)$ with $k < n$. Therefore, it is possible to add l functions $u_1,...,u_l$ with $k+l=n$ such that the differential equations are now:

$$\frac{dp_i}{dt} = P_i,\ \frac{dq_i}{dt} = Q_i,\ i=1,...,k \qquad (2)$$

$$\frac{du_j}{dt} = U_j,\ j=1,...,l. \qquad (3)$$

At this point, the proof of the theorem is almost finished and we refer to Whittaker (see [1] art.116). The key moment of the proof of Lie-Königs theorem is the reduction of a non-exact 1-form to its canonical form. This reduction process amounts to the transformation of a non-exact 1-form $\sum^{n} X_i dx_i$ to another 1-form $\sum^{k} U_i du_i$ with less variables, where $k = n/2$ if n is even, and $k = (n+1)/2$ if n is odd.

In 1992 Volker Perlick was able to prove (see [3]) that a dynamical system when treated in a global perspective admits a Hamiltonian formulation if, and only if, the manifold that supports the given dynamical system has a canonical symplectic 2-form. The global treatment has the advantage of pointing out what is at stake in the Hamiltonization problem by presenting the analytic framework in full detail, placing the hypotheses of an underlying canonical symplectic structure in its key position as the necessary and sufficient condition for the resolution of the question.

For Perlick's global treatment, he starts with a symplectic 2-form written in Darboux coordinates with the main objective of providing a method (practical or theoretical) for obtaining the Hamiltonian of the Hamiltonian formulation. This is done because Perlick (see [3] def. 3) considered a Hamiltonian formulation for a dynamical system to require a pair $<\omega, H>$, where ω is the given canonical symplectic 2-form and H is the Hamiltonian.

Perlick's method consists of obtaining a complete explicit solution to the equations of motion, because if:

$$\omega = \sum_i dp_i \wedge dq_i \qquad (4)$$

we need a solution to the odes in the flow-form:

$$\begin{aligned} \overline{p_i} &= \overline{p_i}(p_i,q_i,t) \\ \overline{q_i} &= \overline{q_i}(p_i,q_i,t) \end{aligned} \qquad (5)$$

to build a map F such that:

$$F^*\omega = \varpi = \sum_{ij}(p_i,q_j)d\overline{p_i}\wedge d\overline{q_j} + \theta \wedge dt. \qquad (6)$$

The parenthesis (p_i,q_j) indicate the Lagrange bracket. If the flow (5) is a canonical flow defined in respect to the underlying canonical symplectic form, we have: $(p_i,q_j) = \delta_{ij}$. Thus the integrability[1] condition over θ is fullfilled and we find

$$\sum_i d\overline{p_i}\wedge d\overline{q_i} + d\overline{H}\wedge dt. \qquad (7)$$

We then obtain a Hamiltonian formulation for the equations of motion defined on a symplectic manifold. Indeed, using Perlick´s terminology, we get an equivalent Hamiltonian formulation as defined in definition 5 of [3].

We see some differences in the methods described above, particularly regarding the global treatment.

[1]A straightforward calculation shows that the integrability condition is

$$\frac{\partial}{\partial t}(p_i,q_j) = 0$$

and is therefore fullfiled in the canonical case.

According to Perlick, we must start with a symplectic 2-form ω and an unknown Hamiltonian H, so that using a map F we get the equivalent Hamiltonian formulation $< F^*\omega, 0 >$. But Whittaker tells us that we must start with a 1-form α and use a map h to transform it into a canonical form $h^*\alpha$.

Locally, however, Perlick's and Whittaker's approaches are related. Given a symplectic 2-form ω, it is clear that if we follow Perlick's treatment, we suppose that ω is written in canonical coordinates. Therefore, we apply the Darboux theorem (for a proof of this theorem see [5] p.132 and p.306 or [17] p. 101) to some 2-form Φ resulting in $F^*\Phi = \omega$. In these coordinates we find, locally, a canonical 1-form β so that $\omega = d\beta$. This is clearly the 1-form obtained using Pffaf´s procedure, because if we apply this procedure to the non-canonical 1-form α we get the canonical 1-form $\beta_0 = h^*\alpha$ from which we must obtain $\omega = d\beta_0$. This means $d\beta = d\beta_0$, hence $\beta = h^*\alpha + c$ with c a constant. As we can see, both approaches must locally give the same results.

The framework used by Perlick and Whittaker to obtain a Hamiltonian formulation for a given vector field consists of:

1. A canonical symplectic structure on the underlying manifold obtained using Darboux or Pffaf reduction.

2. An explicit solution (n integrals of motion) to the dynamical system.

This framework is sufficient to achieve a Hamiltonian formulation for a set of odes. Clearly, the Hamiltonian is both functionally dependent on the set of motion integrals, and is an integral of motion itself as well.

In 1996 Sergio Hojman (see [11]) solved the problem of constructing a Hamiltonian structure starting from the equations of motion without a Lagrangian formulation by using a non-global geometrical setting. This is the Hamiltonization problem, and Hojman solves it using an integral of motion H and a symmetry vector field η of the equations of motion. Thus, if X is the vector field that defines the equations of motion, Hojman then shows that if we have: $[\eta, X] = aX$ or $[\eta, X] = 0$ where $[,]$ is the Lie bracket, we obtain the Hamiltonian formulation. Hojman established the new framework for the Hamiltonization problem as follows: we must use non-canonical coordinates and singular Poisson tensors. There is no reference to Pffaf´s reduction procedure or Darboux's reduction of the non-canonical form to the canonical form. As can be seen, this framework for the Hamiltonization problem is new. The purpose of this paper is to further develop this idea because there are two shortcomings in Hojman´s paper:

a. Hojman does not use modern differential geometric techniques

b. Hojman was not able to generalize his results for $[\eta, X] = a\eta + bX$.

These shortcomings can be applied *mutatis mutandis* to [6,30–32].

If we use modern differential geometric techniques, we are able to generalize Hojman's results thereby obtaining the generalization $[\eta, X] = a\eta + bX$, which had been rejected by Hojman.

Hence this new "minimal framework" of hypotheses is:

1a. The vector field generator of the equations of motion X.

2b. One vector field that forms a Lie algebra with the vector field of the equations of motion.

3c. The use of the axioms for a Poisson manifold (see def. 9, section V, below).

It is worth mentioning that for 3 dimensions, in the work of Goedert, Haas, Hua, Feix and Cairo [30], only hypotheses (1a),(3c) and a given constant of motion are required. Using (1a),(2b) and (3c), we are able to obtain a Hamiltonian formulation for any set of odes in any dimension constructing a constant of motion and a pre-Poisson tensor (see section V def. 9, below). The hypotheses underlying (3c) were known to Lie (see [5] p. 293–294 and [22] p. 237) and every Poisson tensor in canonical form automatically satisfies the hypotheses. Our case is more complex, however, because we are not subject to the rank conditions of the canonical Poisson tensor.

Once we have a clear framework of hypotheses, we must obtain the geometrical framework. Roughly speaking, the geometrical framework of this paper is as follows (for definitions of all concepts see section III):

Every Hamiltonian system defined on the N dimensional manifold M starts from codimension 1 codistribution D^* that locally foliates M with $N-1$ dimensional manifolds given by the condition $H = E$ for all regular values of E. On this basis, the fundamental task is to define a distribution D that locally foliates each leaf of D^* with 1 dimensional leaves . Clearly, D^* is an *a priori* data when the Hamiltonian is known and D is induced by the Poisson tensor P. Therefore, key elements of the Hamiltonian description of dynamics are:

- Codistribution D^*.

- On each leaf of D^* a 1 dimensional distribution D induced by a Poisson tensor P.

These are all the elements needed for a Hamiltonian description, and the Lie-Königs theorem establishes the conditions for reducing D to a canonical distribution D_J when N codimension 1 codistributions D_1^*,..., D_N^* are known through its integral manifolds.

D^* is not given in our treatment; we must obtain it by solving a differential equation. Over D^* we build a Poisson foliation on each one of its leaves using only elements taken from the differential equations themselves in a constructive manner, starting from the less restricted almost-Poisson case (definition 4, section IV below) and moving step by step to the more restricted pre-Poisson manifold case. The final results are found in equations (42a–d) of section V. The solutions to these equations give us: the integral of D^* and a pre-Poisson tensor, which is in effect our new solution to the Hamiltonization problem.

In the framework already outlined, it is not difficult to see the motivation for a converse statement of the symplectic stratification theorem. This theorem can be stated as follows (see [5] p. 302):

Theorem 1.- If P is a finite dimensional Poisson manifold, then P is the disjoint union of its symplectic leaves. Each leaf in P is an injectively immersed Poisson submanifold and the induced Poisson structure on the leaf is symplectic. The dimension of the leaf through a point z equals the rank of the Poisson structure at that point.

Hence a converse statement implicitly suggests: each foliated manifold admits a Poisson structure on each leaf. We shall prove this claim in section V.

3. Theoretical Framework

The objective of this section is to establish some important definitions and terminology. We wish to point out that our definition of distribution is taken from Souriau [7] p. 38. Chevalley in ref [13] p. 86–87, calls such an assignment a distribution of contact elements, a name with some appeal, while Berndt in [4] p. 57 coins the name c-dimensional differential system. A brief discussion of the equivalence of several definitions of the concept of foliation can be found in [20]. We agree with Souriau on the first three definitions, but not on his notation.

We also point out that our definition 4 is contained in Oppenheimer´s *Festschrift* issue [12] p. 575 of 1964 in a paper by Res Jost. In the non-holonomic mechanics literature, for example [21] p. 1382, the object introduced in definition 4 is called an almost-Poisson structure. We follow this terminology for the most general case of non-injectivity and non-surjectivity of the tensor introduced (see [5] p. 57), but we use the term "Jost´s manifold" for the case when the tensor is injective and surjective. Res Jost was one of the first who observed that the closedness of the symplectic form is equivalent to Jacobi´s identity for the Poisson tensor [15] p. 405. Incidentally, it is not common to distinguish the Poisson tensors by rank conditions.

Def 1. Given a smooth N dimensional manifold M, we say that the map D is a k-dimensional "distribution" if, and only if, for each $p \in M$ we have $D(p) \subset T_pM$, where T_pM is the tangent space of M at p.

D defines a map of the form $D : U \subset M \to D(p) \subset T_pM$ and with the help of a distribution we attach a subspace of TM to each point of M in a smooth manner if each of the elements of D (tangent vectors) is smooth.

There is an integer p such that $p + k = N$ known as the codimension of D. The dimension of D is its rank and may change at each point because its base vectors may become linearly dependent, but we shall suppose that it is constant unless otherwise stated.

Def 2. We say that $\Theta \subset M$ is an "integral manifold" of D if, and only if, for every $p \in \Theta$ we have $T_p\Theta = D(p)$.

Def 3. We say that the correspondence $p \to D(p)$ is a "foliation" if, and only if, every $p \in M$ belongs to an integral of D.

If D is a foliation of M with integrals $\Theta_1, ..., \Theta_n$, using definition 3 we can therefore express M as the following disjoint partition:

$$M = \bigcup_{i=1}^{n} \Theta_i. \tag{8}$$

Clearly, a 1-dimensional foliation on a manifold $M = \mathrm{Re}^N$ is a smooth assignment to each of its points p of a vector $X(p)$. We then have: $D(p) = span\{X(p)\} \subset T_p\mathrm{Re}^N$.

If local coordinates of Re^N are $< x_1, ..., x_N >$, the 1-dimensional foliation $span\{X(p)\}$ is defined as the following set of differential equations:

$$\frac{dx_i}{dt} = X_i(x_1, ..., x_N),\ i = 1, ..., N. \tag{9}$$

Any solution of (9) is a smooth path γ obtained with the help of an initial condition selected on a transversal path to γ and parametrized by a 1-dimensional coordinate t. We write $\gamma(b_1, ..., b_N, t) = \gamma(b, t)$ to note the solution path selected by the initial condition b and parametrized by t. For any constant $t = t_0$, the variable values of $\gamma(b, t_0)$ generate a path transversal to the solution path generated for constant $b = b_0$ and variable t. Thus we can write the following decomposition for the manifold Re^N as a result of the 1-dimensional distribution $span\{X\}$:

$$\mathrm{Re}^N = \bigcup_{b\in I_1} \bigcup_{t\in I_2} \{\gamma(b, t)\} \tag{10}$$

where $I_1 \subset \mathrm{Re}^N$ and $I_2 \subset \mathrm{Re}^2$. The $\bigcup_{t\in I_2} \{\gamma(b, t)\}$ union gives us a complete 1-dimensional leaf of $\{X\}$ which

[2]"Re" means "Real line", hence, obviously, Re^N is a cartesian product.

we may call $\Theta(b)$. These leaves are clearly integrals of D. Therefore Re^N is the disjoint union of each leaf of $span\{X\}$ as represented in (10).

Def 4. V_N is a N-dimensional almost-Poisson manifold [22] if, and only if, M and P exist such that:

i $V_N =< M, P >$

ii M is a smooth n-dimensional manifold.

iii P is a smooth singular skew-symmetric 2-contratensor.

This weak definition can be made stronger by replacing (iii) with the following supposed condition (we will call it (iii*)).

iii* P is a smooth non-singular skew-symmetric 2-contratensor.

If we use (iii*) instead of (iii), we get a Jost´s manifold V_N^*. Every Jost´s manifold is of even dimension, a restriction not shared by almost-Poisson manifolds.

Def 5. D^* is the dual distribution -or codistribution- of D if, and only if:

i A smooth inner product is defined on the bundle $TM \times TM$ denoted by:

$$g : T_pM \times T_pM \to \mathrm{Re}\,.$$

ii Every element of D^* is the result of the contraction of g with an element of D:

$$D^* = \{\omega \mid \iota_X g = \omega, X \in D\}.$$

iii For every $X \in D$ and every $\omega \in D^*$ we have $\iota_X\omega = 0$.

If the dimension of D is k, then the dimension of D^* is $p = N - k$ which is the codimension of D. So, the codimension of D^* is k.

Not surprisingly, the elements of D^* are 1-forms or co-vectors, so, in fact, we have enriched the geometrical framework with the use of a Riemann manifold with metric g, and the inner products are of the form:

$$\iota_Y\omega = \iota_Y(\iota_X g) \tag{11}$$

This is a double contraction with the metric tensor.

In addition, when we are on an almost-Poisson manifold V_N, we have a skew-symmetric 2-contratensor$P : T^*M \times T^*M \to \mathrm{Re}$ defined over cotangent spaces of covectors. When we contract P with a covector, we get a vector which defines a map $T^*M \to TM$. If this map is an isomorphism, we have a Jost´s manifold. If not, we have an almost-Poisson manifold.

Def 6. A foliation is induced by a 2-contratensor P if its leaves are integrals of the following vector field:

$$\iota_\omega P \tag{12}$$

where $\omega \in T^*M$ and ι the contraction.

On almost-Poisson or Jost manifolds, we can induce foliations by the tensor P.

Every element of a codistribution D^* allows us a different kind of foliation of the underlying manifold M because, as is well known, when we can integrate the differential equation $\omega = 0$, we get $\omega = df$. So, using condition (iii) of definition 5, we know that the vector field on the distribution D is tangent to the $N-1$ dimensional manifold $\ker_a^{N-1} f = \{x \in M \mid f(x) = a\}$. Thus this manifold is an integral, according to definition 3 of distribution D, giving us another decomposition of M dual to that of D.

Therefore, with the help of codistribution D^* we find:

$$M = \bigcup_a \ker_a^{N-1} f \tag{13}$$

and with the help of D we obtain the result:

$$\ker_a^{N-1} f = \bigcup_b \Theta^a(b). \tag{14}$$

The use of Θ^a is required because when we change level a, we change the initial conditions that define the leaf Θ. We shall introduce more definitions in the following section, but only where they are required for specific points.

4. The Almost-Poisson Case

In this section we prove some very simple results for Hamiltonization in almost-Poisson manifolds. The results of this section are not the solution to the Hamiltonization problem because we do not introduce the important Jacobi´s identity, but rather, describe the steps which lead to the solution. Naturally, some elements of our construction may lack a precise determination due to the theory conditions because an almost-Poisson manifold is a very weak framework and many elements cannot be determined.

Lemma 1. *Given a codimension 1 integrable codistribution D^* obtained by a l-differentiable function $f \in C^l(V_N, Re)$ of maximal rank on an almost-Poisson manifold V_N, we have a foliation of $\ker_a^{N-1} f$ induced by the skew-symmetric 2-contratensor P.*

Proof. On our Riemann manifold M, the set $\ker_a^{N-1} f$ is a smooth $N-1$ dimensional manifold [9]. Any path γ is $\gamma \in \ker_a^{N-1} f$ if, and only if, its tangent vector $\gamma_* d/dt$ is such that:

$$\iota_{df}\left(\gamma_* \frac{d}{dt}\right) = 0. \tag{15}$$

If this condition is fullfilled, the path γ is a submanifold of $\ker_a^{N-1} f$, hence its vector field generator must be zero when contracted with f. It is a matter of straightforward calculation to show that if the vector field which generates γ is given by $\iota_{df} P = \gamma_* X = \gamma_* d/dt$, we get $\iota_{df}(\iota_{df} P) \equiv 0$ from (15). ■

If we use coordinates over $V_N = < \mathrm{Re}^N, P >$ the meaning of lemma1 becomes clear because condition (15) is expressed as:

$$\sum \frac{dx_i}{dt}\frac{\partial f}{\partial x_i} = 0$$

If the vector field is given by

$$\frac{dx_i}{dt} = \sum P^{ik}\frac{\partial f}{\partial x_k},$$

it is a matter of substitution to verify the lemma (for comparison see [31]).

Foliation is induced by a tensor P which we introduce arbitrarily by means of an almost-Poisson manifold. There is no particular reason to avoid the introduction of P, but we would require compelling reasons to do so.

Lemma 2. *Given a 1-dimensional distribution D, its underlying manifold M can be made an almost-Poisson manifold V_N if the following sufficient conditions are met:*

i *The existence of an integral f of codistribution D^*.*

ii *The existence of a distribution D_n which doesn't have common integrals with D.*

Proof. The existence of f given a codistribution D^* is always locally guaranteed because f is the solution of a partial differential first order equation and a power series solution can be made convergent on a given disk.

Therefore, we can locally build the tensor:

$$P = -\frac{X \wedge n}{\iota_{df} n} \tag{16}$$

which can be used as:

$$\gamma_* \frac{d}{dt} = \gamma_* X = -\gamma_* \left(\frac{(\iota_{df} X) n}{\iota_{df} n} - \frac{(\iota_{df} n) X}{\iota_{df} n} \right). \tag{17}$$

The expression on the right hand side is $\iota_{df} P$, as a result we get:

$$\gamma_* \frac{d}{dt} = \gamma_* \iota_{df} P \tag{18}$$

and the distribution is induced by a skew-symmetric 2-contratensor using only the stated conditions. ■

In lemma 2 we have introduced a given almost-Poisson manifold, constructed out of D, D^*, D_n. So, its construction is based on the vector field generator X, on the arbitrary element f, and on the generator of D^* denoted by n.

The method used in the proof of lemma 2 was first used by Hojman on the Poisson case. As we shall see in the next section, all arbitrary elements in the tensor (16) construction are eliminated. Therefore, Hojman´s theory is strong enough to determine f and n.

Clearly, any differential equation with a first integral of motion can be reduced to the form (18) if the conditions stated in lemma 2 are satisfied. This first integral of motion is the Hamiltonian of a Hamiltonian formulation. In this way, if we can find distribution D_n (not a difficult task), then every distribution D can be used locally to build an almost-Poisson manifold V_N with distribution D_P defined by (18).

We now introduce an almost-Poisson bracket.

Lemma 3. *The necessary condition for a manifold $O =_{def} \ker_a^{N-1} f \cap \ker_c^{N-1} g$ to accept a common almost-Poisson distribution D_P is:*

$$\iota_{dg}(\iota_{df} P) = 0. \tag{19}$$

Proof. If γ is a leaf of $\ker_a^{N-1} f$, we can write equation (15) for f, and if $\gamma \in \ker_c^{N-1} g$ we have:

$$\iota_{dg}\left(\gamma_* \frac{d}{dt}\right) = 0 \tag{20}$$

for g. Using lemma1, we write $\gamma_* d/dt = \iota_{df} P$, and are then able to write equation (20) as

$$\iota_{dg}\left(\iota_{df} P\right) = 0 \tag{21}$$

which creates a new condition to be satisfied. This condition is necessary because if it is not fullfilled, the path cannot be contained in O ■

Therefore, on our almost-Poisson manifold we can introduce almost-Poisson brackets as a definition:

Def 7. $(f, g)_J \in C^k(V_N, \mathrm{Re})$ is an almost-Poisson bracket of f and g if, and only if:

$$(f, g)_J = \iota_{df}(\iota_{dg} P)$$

Def 8. An almost-Poisson vector field X_f is given by:

$$\iota_{dg} X_f = (g, f)_J \tag{22}$$

If f is globally defined, then X_f is as well. Otherwise, it is locally defined. Almost-Poisson brackets are just one step in our treatment; they don´t satisfy Jacobi´s identity, which is essential for any bracket [8].

Lemma 3 can be generalized to the case of p-functions $f_1, ..., f_p$ as defined on an N-dimensional almost-Poisson manifold. Using the same reasoning as in lemma 3, we can write the necessary conditions for a manifold to accept a common almost-Poisson distribution as follows:

$$(f_i, f_j) \mid_{\bigcap_{i=1}^{p} \ker f} = 0,\ i,j = 1,...,p \tag{23}$$

If the map

$$F =< f_1, ..., f_p >: V_N \to V_p \tag{24}$$

is of maximal rank, $\min\{N, p\}$, we can write down:

$$(f_i, f_j) = \sum_k c_{ij}^k f_k \tag{25}$$

as shown by Olver in [8] proposition 2.10, p. 82.

5. The Poisson Case

The objective of this section is the construction of the generalization of Hojman´s procedure to obtain Hamiltonian formulations for arbitrary odes. To do this we must introduce Poisson manifolds as a logically stronger framework of hypotheses than an almost-Poisson manifold, as well as give a set of transformations that would leave the Poisson manifold invariant (see [5,8,10,16,26]). Such transformations are known as Poisson, or canonical, transformations.

However, the transformation theory is not an inmediate result of Poisson's manifold definition because it only appears when certain conditions of consistency or of locality based on Jacobi´s identity are used. We demonstrate this later on in this section using Marsden and Ratiu's [5] presentation of Poisson manifolds and Souriau's [7] presentation of symplectic manifolds.

Next in our theory, we start with the construction of a pre-Poisson manifold found in lemma 4, then construct the invariance theory in lemmas 5, 6 and 7. The construction of a pre-Poisson manifold does not imply a transformation theory since the notion of a pre-Poisson manifold and the construction of the related transformation theory are independent conditions. Finally, as we shall see, these constructions are enough to obtain the generalization of Hojman´s results in the form of equations (42a–d).

A closer look at Marsden-Ratiu ´s proof of invariance ([5] p. 299–300) reveals that their reasoning involves the use of consistency conditions from which the truth of the theorem´s proposition arises.

Marsden-Ratiu start with:

$$\begin{aligned}\iota_{dG}\left(\iota_{dF} P\right) =_{def} (F, G)& \\ = X_G(F) = \text{Poisson´s bracket}&\end{aligned} \tag{26}$$

therefore:

$$\pounds_{X_H}[(F,G)] = X_H(F,G) = ((F,G),H) \tag{27}$$

These equations are definitions which do not involve theory postulates. They then write:

$$\begin{aligned}\pounds_{X_H} P(dG, dF) = (\pounds_{X_H} P)\,(dG, dF)& \\ + ((F,H),G) + (F,(G,H))&\end{aligned} \tag{28}$$

as another identity. However, by using Jacobi´s identity (a key theoretical postulate), on the last two terms of (28), they obtain the result:

$$\begin{aligned}\pounds_{X_H} P(dG, dF) = (\pounds_{X_H} P)(dG, dF)& \\ + ((F,G),H)&\end{aligned} \tag{29}$$

If the theory is consistent, as we suppose it is, then (27) and (29) are equal, but only if $\pounds_{X_H} P = 0$. Thus, Jacobi´s identity implies invariance if we want a consistent theory. Taking this into consideration, the Poisson manifold notion and the transformation theory cannot be independently constructed if we want a consistent theory.

On symplectic manifolds, the invariance of the 2-form is linked with conditions that are usually taken for granted. For symplectic invariance, as proved by Souriau in [7] as theorem 9.21, we see that, given a Hamiltonian vector field X_H and the symplectic 2-form Ω, we have in general:

$$\pounds_{X_H}\Omega = d\iota_{X_H}\Omega + \iota_{X_H} d\Omega. \tag{30}$$

Consequently, using Jacobi´s identity, $d\Omega = 0$, we can write:

$$d\iota_{X_H}\Omega = 0 \tag{31}$$

for invariance. This condition is fulfilled when $\iota_{X_H}\Omega = -dH$, that is, when X_H is a globally defined vector field, (see for example, [18] p. 130). However, this global definition depends on the disappearance of the first DeRham cohomology class $H^1_{DR}(M, \mathrm{Re}) \ni [\iota_X \Omega] = 0$ of the manifold M with coefficients in Re. Hence, in general, we can only speak of invariance in regard to locally defined Hamiltonian vector fields. We can see in this case that the notion of a symplectic manifold and the transformation theory can be independent. In other words, we may have $d\Omega = 0$ locally but globally, for topological reasons, $\iota_{X_H}\Omega$ is not an exact form.

We have therefore learned that we can obtain an invariance theory, but not as an automatic result of the notion of a Poisson manifold. For our theory, we introduce the notion of a pre-Poisson manifold in definition 9, but require the transformation theory obtained in lemmas 5,6,7 and 8. This results in equations (42a–d) which will represent the conditions of our theory and the generalization of Hojman´s equations (A) and (B).

We now briefly discuss definition 9 to establish the reasons of our terminology. Marsden and Ratiu in [5] p. 286, Olver in [8] p. 390–391 and Guillemin and

Sternberg in [26] p. 89 present the concept of the "Poisson manifold" by simply introducing a Poisson bracket on the ring of functions over a given manifold. This is along the lines of Lie and Engel in [23] p. 234 but without the use of any rank conditions on the tensor. Of course, at the time of Lie and Engel many modern concepts were not in existence, for example what they called "Functionengruppen" is now known as "Lie algebra". The name "Poisson manifold" was coined by Lichnerowicz [24].

However, we need to distinguish between a bijective, non-singular case and a non-injective, singular case, even if the latter is more general than the former. For the singular case in the symplectic framework, Souriau in [7] p. 82, proposes the name "pre-symplectic". This is why we propose the name "pre-Poisson", which is short for a non-injective Poisson tensor.

Def 9. Po^N is a N-dimensional pre-Poisson manifold if, and only if, M and P exist such that:

i $Po^N =< M, P >$.

ii M is a N-dimensional smooth manifold.

iii P is a skew-symmetric singular 2-contra-tensor defined on $T^*M \times T^*M$.

iv The Schouten commutator of P with itself is zero.

We shall call P "pre-Poisson tensor".

Def 10. A vector field X is a Hamiltonian vector field if, and only if, ω and P exist such that:

i $\omega \in T^*M$.

ii P is a pre-Poisson tensor.

iii $\iota_\omega P = X$.

If $d\omega = 0$, we have a "locally defined Hamiltonian vector field", and if $\omega = dH$ all along the manifold, we have a "globally defined Hamiltonian vector field". Both cases are denoted by X_H and we use the terms "locally defined" or "globally defined" to distinguish them.

Using definition (9), we obtain the most general conditions which allow us to write down:

$$P = -\frac{X \wedge n}{\iota_{df} n} \tag{16}$$

is a pre-Poisson tensor where f is an integral of D^* in lemma 4. However, it is not invariant in front of X, which is a locally defined Hamiltonian vector field. We show how to fill this gap in lemmas (5) and (6).

Lemma 4. *$P = -\frac{X \wedge n}{\iota_{df} n}$ is a pre-Poisson tensor, and each of the leaves of D is a pre-Poisson manifold, if, and only if, X and n are the basis of a 2-dimensional smooth integrable distribution D_D defined on each leaf of D^*.*

Clearly $D \subset D_D$, so, every integrable 2-dimensional distribution has a 1-dimensional distribution defined over a pre-Poisson manifold with pre-Poisson leaves. This is an equivalent formulation of lemma 4.

Proof. We must prove that if the Schouten bracket of P with itself is zero, then $[X, n] = aX + bn$.

To do this, we must compute the Schouten commutator. After a straightforward calculation following [5] p. 31İ, the expression is:

$$2(X \wedge n) \wedge [X, n]. \tag{32}$$

Hence, if D_D is integrable, the Schouten bracket is zero by associativity. To prove the second part, we need to show that (32) can only be solved by the integrability condition $[X, n] = aX + bn$. This can be seen as follows: by definition, X and n are linearly independent, so, (32) shows by associativity (see [9], p. 208) that $[X, n]$ must be linearly dependent on X and n. ■

In the proof of lemma (4), we have not used the invariance of P in front of X in any step. Therefore, it is an independent proposition that must be proven.

The proof in this case is somewhat complex because it involves the consideration of consistency between a pair of first order partial differential equations. We first establish the conditions of invariance (lemma 5). After, we show how to satisfy them (lemma 6). When we are trying to solve sets of simultaneous partial differential equations of order 1, the key concept is that of "involution", as used, for example, by Goursat in [25] p. 274. But this concept and its use by Goursat require an underlying canonical Poisson structure which we do not need in our treatment.

Lemma 5. *The Poisson tensor $\frac{X \wedge n}{\iota_{df} n}$ is invariant in front of the flow generated by X if and only if:*

$$\pounds_X \left(\frac{1}{\iota_n df} \right) = -\frac{b}{\iota_n df}. \tag{33}$$

Proof. The Lie derivative of P (the sign in equation (16) is of no importance here) is given by:

$$\pounds_X \left(\frac{X \wedge n}{\iota_n df} \right) = \pounds_X \left(\frac{1}{\iota_n df} \right) X \wedge n + \frac{1}{\iota_n df} X \wedge [X, n], \tag{34}$$

where the second term of (34) becomes simply $b\frac{X \wedge n}{\iota_n df}$. Hence, the Lie derivative is:

$$\pounds_X \left(\frac{X \wedge n}{\iota_n df} \right) = \left(\pounds_X \left(\frac{1}{\iota_n df} \right) + \frac{b}{\iota_n df} \right) X \wedge n. \tag{35}$$

Accordingly, if (33) is correct, the Poisson tensor is invariant in front of the flow. If (35) is zero, then (33) is correct. ■

To solve (33), we propose using the following differential equation:

$$\iota_n df = k,\ C^l \ni k \neq 0. \tag{36}$$

We need to start by determining k independently. We have another differential equation for f given by $\iota_X df = 0$. Consequently, we must show its consistency with (36) using the Lie algebra $[X, n] = aX + bn$. We now show that the consistency condition defines k, and what´s more, this condition over k is the condition of invariance (33).

Lemma 6. *The pair of differential equations:*

$$\iota_X df = 0, \tag{37a}$$
$$\iota_n df = k \tag{37b}$$

is consistent, if and only if, k solves the following differential equation:

$$\iota_X dk = bk. \tag{38}$$

Proof. For a finite k we have:

$$\iota_{[X,n]} df = b\iota_n df = kb \tag{39}$$

from Schouten bracket conditions and the identity:

$$\iota_{[X,n]} df = \pounds_X (\iota_n df) - \iota_n \pounds_X (df) \tag{40}$$

which can be reduced to[3]:

$$\iota_{[X,n]} df = \pounds_X (k) = \iota_X dk \tag{41}$$

by using (37a–b). For consistency, (39) and (41) must be equal. So, if k solves (38), system (37a–b) is consistent. If it is consistent, then (39) and (41) must be equal. ■

Lemma 7. *The condition of invariance (33) is equivalent to the condition of consistency (38).*

Proof. Clearly:

$$\pounds_X (\frac{1}{\iota_n df}) = \iota_X d\left(\frac{1}{\iota_n df}\right) = -\frac{1}{k^2}\iota_X dk = -\frac{b}{k},$$

using only lemma 5 and common identities. ■

Now we have all the elements to write the following theorem:

Theorem 1. (generalized Hojman) *P given by (16) is a Poisson tensor and X an infinitesimal generator of Poisson transformations if, and only if, the following conditions are satisfied:*

$$[X, n] = aX + bn, \tag{42a}$$
$$\iota_n df = k, \tag{42b}$$
$$\iota_X df = 0, \tag{42c}$$
$$\iota_X dk = bk. \tag{42d}$$

Proof. Because of lemma 4, P is pre-Poisson tensor. Using lemma 6, the pair of differential equations is consistent and solves the condition given by lemma 5. ■

Hojman originally considered two cases [11]:

i $[X, n] = 0$, $\iota_X df = 0$.

ii $[X, n] = aX$, $\iota_X df = 0$.

He rejected the term bn because it is not consistent with his theory´s postulates. This is the case, of course, if lemmas 5 and 6 are not used.

Case (ii) was treated by Lie, showing that the set of odes is integrable by quadratures. If D is a distribution generated out of X, then a codistribution D^* generated by 1-form ω such that $\iota_X\omega = 0$ is an equivalent formulation for the set of odes. Lie then showed that if the algebraic condition (ii) is correct, the 1-form ω is integrable with integrability factor $\frac{1}{\iota_n\omega}$ (see [14] for a generalization, and [10] for a classical statement). Both cases are clearly consistent, as the following calculation shows:

$$\iota_{[X,n]} df = a\iota_X df = 0 \text{ or } \iota_{[X,n]} df = 0$$

as Schouten bracket conditions. Equation (40) is:

$$\iota_{[X,n]} df = \pounds_X(\iota_n df)$$

Consistency requires that $\pounds_X(\iota_n df) = 0$, which demonstrates that $\iota_n df$ is also a constant of motion, as shown by Hojman.

Lemma 6 gives us another condition due to differentiation. This new condition is equation (38) which is added as a constraint to form system (42a–d). So the obvious questions are whether system (42a–d) is consistent and whether more derivations can produce more conditions to be satisfied, or simply produce identities. Indeed, we find the system is consistent and that more derivations produce identities.

Lemma 8. *The system of equations (42a–d) is consistent and more derivations produce identities.*

Proof. We take an exterior differentiation on (42b), thereby:

$$dk = d\iota_n df = \pounds_n(df).$$

Using this result on (42d), we write down:

$$\iota_X \pounds_n(df) = b\iota_n df.$$

Hence, using known equations, [5] p. 128, we get:

$$\iota_{[X,n]} df + \pounds_n(\iota_X df) = b\iota_n df.$$

Finally, we use (42a) and (42c) to obtain the identity:

$$b\iota_n df = b\iota_n df.$$

[3] On the second term of (40), we apply Cartan´s formula to write

$$\pounds_X (df) = d(\iota_X df)$$

which is zero because of (37a)

■

As we can see, system (42a–d) does not produce new equations by differentiation. What we must do now is solve this system of equations.

We have all the elements for the following:

Theorem 2. (Local inversion of the symplectic) stratification theorem *Let M be an N-dimensional smooth manifold stratified according to*

$$U_\alpha = \bigcup_{a\in I_\alpha} \ker_a^{N-1} f$$

on each local chart U_α. Here we have an interval of regular values $I_\alpha \subset Re$ for each α. So, each leaf $\ker_a^{N-1} f$ is a Poisson manifold.

Proof. From lemmas 1 and 2, we get an almost-Poisson manifold by taking $\frac{X \wedge n}{\iota_{df} n}$ and using theorem 1 to get the required Poisson manifold which imposes its conditions on X and n. ■

6. A Solution to the Hamiltonization Problem

We now have the full set of consequences of the minimal framework that can be useful to get a solution to the "Hamiltonization problem" within its boundaries. Hence the following definition is useful:

Def 11. HS is a Hamiltonian system if, and only if, Po^N and H exist such that:

i $HS =< Po^N, H >=< M, P, H >$.

ii Po^N is a N-dimensional smooth pre-Poisson manifold.

iii H is a smooth real valued function defined on Po^N.

iv $\iota_{dH} P = X_H$, where X_H is, by definition, a local Hamiltonian vector field if H is locally defined. Otherwise, it is a global Hamiltonian vector field.

In the approaches usually taken, the question is how to get the Hamiltonian function assuming there is an underlying canonical symplectic structure. This question is answered, as seen in section II, using an explicit solution to the odes and constructing a transformation theory to supply the Hamiltonian function.

The overall approach indicated by the minimal framework suggests that the solution of equations (42a–d) must be the starting point. So, we must determine how to obtain the Poisson structure and the Hamiltonian. If we achieve this, we can avoid the necessity of Pffaf´s and Darboux´s reduction procedures because we are not restricted by the *a priori* supposition of a canonical symplectic form and the necessity to reduce every tensor to this canonical form.

Once we have obtained theorem 1 by using only minimal framework postulates, the Hamiltonization problem involves finding a solution to system (42a–d) with given vector field X.

Clearly, solving this system with given X produces a Hamiltonian system according to definition 11:

$$HS_X = \left\langle M, \frac{X \wedge n}{\iota_n df}, f \right\rangle$$

We have thus proved that HS_X is actually a Hamiltonian system. With any differential equation, we can associate a triple HS_X so that its vector field becomes a Hamiltonian vector field according to definition 11. As we can see, solving the Hamiltonization problem simply involves finding a solution to (42a–d) given X. Therefore, any transformation based on explicit solutions to the odes is not necessary.

We now give an example of the construction of a new Poisson structure for the state space Re $\times S^1$of a harmonic oscillator, which in local coordinates is $< t, r\cos\theta, r\sin\theta >$ with $r = cte$. The path given by $\theta = \theta(t)$ along the cylindrical state space forms a helix.

Vector field generator on Re^3 is:

$$X = \frac{\partial}{\partial t} + \omega\left(x\frac{\partial}{\partial v} - v\frac{\partial}{\partial x}\right). \tag{43}$$

We shall put $L = x\frac{\partial}{\partial v} - v\frac{\partial}{\partial x}$ and propose:

$$n = \frac{\partial}{\partial t} + n_1\frac{\partial}{\partial v} + n_2\frac{\partial}{\partial x}. \tag{44}$$

Commutation rules (42a) give us:

$$Ln_1 - n_2 = \frac{b}{\omega}n_1 + ax, \tag{45}$$

$$Ln_2 + n_1 = \frac{b}{\omega}n_2 - av, \tag{46}$$

$$a = -b. \tag{47}$$

Under a coordinate change Re$^2 \to$ Re $\times S^1$, defined by $x = r\cos\theta$ and $y = r\sin\theta$ with $r = cte.$,we obtain:

$$\begin{aligned} n_r &= 0 \\ n_\theta &= n_1\frac{\partial\theta}{\partial v} + n_2\frac{\partial\theta}{\partial x} \end{aligned}$$

for the vector components of n at the new coordinate cover. Equations (45),(46) become the single equation:

$$\frac{\partial n_\theta}{\partial\theta} = \frac{a}{\omega}(\omega - n_\theta) \tag{48}$$

with the solution:

$$n_\theta = \omega - \varphi(r^2)\exp\left(-\left(\frac{a}{\omega}\right)\theta\right). \tag{49}$$

For n at the original coordinates, we then easily obtain:

$$n = \frac{\partial}{\partial t} + eL, \quad (50)$$

where $e = \omega - \varphi(x^2 + v^2)\exp\left(-(a/\omega)\arctan(v/x)\right)$. Now, we must obtain the Hamiltonian H and the function k as solutions to:

$$\frac{\partial H}{\partial t} + \omega LH = 0,$$
$$\frac{\partial H}{\partial t} + eLH = k,$$
$$\frac{\partial k}{\partial t} + \omega Lk = -ak.$$

The solutions are:

$$k = \psi(x^2 + v^2)\exp(-at), \quad (51)$$

$$H = -\frac{\omega}{a}\frac{\psi(x^2 + v^2)}{\varphi(x^2 + v^2)} \times \exp\left(-at + \frac{a}{\omega}\arctan\left(\frac{v}{x}\right)\right), \quad (52)$$

as verification shows. Finally, the Poisson tensor is:

$$P_{ha} = \frac{\omega - e}{k}\frac{\partial}{\partial t} \wedge L = \frac{\varphi(x^2 + v^2)}{\psi(x^2 + v^2)} \times \exp\left(at - \frac{a}{\omega}\arctan\left(\frac{v}{x}\right)\right)\frac{\partial}{\partial t} \wedge L. \quad (53)$$

Therefore, we have proved the following proposition:

Proposition 1. *The triple:*

$$< Re \times S^1, P_{ha}, H >$$

is a Hamiltonian system.

7. Conclusion

In an attempt to find a new solution to the Hamiltonization problem, we have worked out the hypotheses of the newly introduced theory called "minimal framework" in detail, and have obtained a new formulation for the classical Hamiltonization problem of Lie-Königs with less restricted assumptions, thereby generalizing previous results obtained by Hojman.

As the simple example of the harmonic oscillator shows, the operative setting of the main equations deduced to get Hamiltonian formulations is useful when we are able to solve equations (42a–d). Moreover, the theoretical hypotheses are useful for proving some theorems in a straightforward manner.

Perlick constructed his map $\omega_0 \to < F^*\omega_0, 0 >$ from the set of symplectic structures onto the set of Hamiltonian formulations using his theoretical setting and the explicit solution to the equations of motion F. This was useful to determine the Hamiltonian of his theory. In our case, we take 2-dimensional integrable distributions and map them to Poisson structures and then to Hamiltonian formulations. In other words, as a result of our theory, the set of integrable 2-dimensional distributions over M, $ID_2(M)$ is mapped to the set of decomposable Poisson tensors over M, $DPT(M)$, which is clearly not void. Hence, we have:

$$\Im : ID_2(M) \to DPT(M)$$

The meaning of the mapping is clear if we define $ID_2(M)$ as a category [27]. Its objects are the 2-dimensional integrable distributions, and its arrows are a subgroup of the diffeomorphisms of M lifted to TM. The objects of $DPT(M)$ are sets with at least one Poisson tensor while the arrows are the Poisson maps. $\Im$ can then be read as a functor whose construction is the main result of this paper.

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