PACS №: 02.10.Ud; 04.20.Cv; 04.50.+h; 95.36.+x

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A Comprehensive Theory of Gravitation from START

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Abstract

Gravitation can be casted as a physical theory by itself. The steps are: to define the interacting bodies, to define the interaction and its properties, to define the energies and forces involved, to show the self consistency of the mathematical structure involved, and a discussion of the (proportionality) relation between gravitational and inertial mass. The mathematical frame for this development is a 5-D quadratic space reference manifold: space, time and action (START). The principles, that is concepts defined through their use in the theory, are presented.

1. Gravitation

Gravitation is an example of a physical theory where the description of nature is based both on direct observation of phenomena as seen by the observer and a collection of theoretical variables which can be either semi-intuitive (as the concept of force) or totally theoretical (as the concept of work) or primitive as space, time, material objects, system where the use defines the concept itself.

Gravitation also occupies a particular position in physics because historically we have used either kinematical relations as 'free fall' (Galileo) or statics relations as 'gravitational force'. Otherwise the primitive concepts are defined in a tautological cycle: space is related to the locii of material objects, material objects are defined as that which occupies a volume in space, and time is perceived as a parametrization of the succession of changes for the material objects and space. For a mathematical theory of gravitation the best point of departure is to define a manifold describing these quantities.

As mentioned in the abstract our presentation will cast Gravitation as a physical theory by itself. For this purpose, after defining the basic manifold Space-Time-Action, the steps are: to define the interacting bodies, to define the interaction and its properties, to define the energies and forces involved, to show the self-consistency of the mathematical structures involved, and a discussion of the (proportionality) relation between gravitational and inertial mass. This is the procedure to be followed below.

2. Gravitation from START

The acceptance of space and time as concepts parametrized by a 4-D manifold (x, y, z, t say) is now universal in physics. Otherwise material bodies (and radiation for that purpose) are always introduced as additional concepts. Matter and radiation are presented mathematically as a density of energy in space, a non covariant definition which transforms into a density–current of energy, that is energy–momentum.

The procedure we will follow in this paper is to use a scalar quantity a ACTION and its density α in space–time. For a stationary matter–radiation distribution the space–time density of action is trivially equivalent to a density of energy $\varepsilon/\Delta x\Delta y\Delta z$ in space for a given observer

$$\alpha = \frac{a}{\Delta x\Delta y\Delta z\Delta t} \underset{\text{stationary system}}{=} \frac{\varepsilon\Delta t}{\Delta x\Delta y\Delta z\Delta t} = \frac{\varepsilon}{\Delta x\Delta y\Delta z}. \quad (1)$$

To define a space–time–action 5-D manifold. The formal procedure to integrate space and time was to define the quadratic space of Minkowski, in our case we will introduce a 5-D quadratic form, a generalization of the quadratic form which historically started with the Pythagorean formulation. Schematically, Table 1, where we are defining Table 2, here l, (x, y, z), c, h, w, a, $E_{(0)}$ are distance, distance components, vacuum speed of light, Planck's constant, distance equivalent to action, action of a system and characteristic energy of the system respectively.

There is a nesting from {S-T-A (general manifold) with quadratic form $QForm$ which induces a Clifford Algebra $Cl_{1,4}$} when fixing (Observed system W + Observer O) to obtain the W-system's experimental description in O-observer's space, Table 3.

Once this geometry is defined Physics is introduced through the: **START Relativity Principle**

> • **"All trajectories describing physical objects are null for all observers"**
> • **"The vacuum Speed of Light is c for all observers"**

and the: **START Description Principle**

> • **"Physical objects are described by bundles of null trajectories"**
> • **"The environment of physical objects is described by interaction potentials and chemical potentials"**

The null trajectories condition reduces the five dimensions to effective four dimensions.

A non-zero chemical potential allows the physical object to exchange energy with the environment. The interaction potentials describe energy-momentum being shared.

Our geometrical structure allows the development of a new, deductive scheme, formalism. For this purpose we need a minimal set of FUNDAMENTAL PRINCIPLES and POSTULATES, in such a form to obtain a comprehensive theory for Physics:

START Relativity (the laws of physics are invariant under a 5-D Poincaré group which includes 5-D Lorentz transformations);

Existence (physical objects are represented by energy densities);

Least Action (physically acceptable 'trajectories' correspond to null, optimal possible, trajectories in START);

Quantized Exchange of Action (we can define systems or subsystems as those among a quanta of action can be exchanged) and,

Choice of Descriptions (we should allow all useful physical models to be employed and properly based in START).

2.1. Action Density and its partial derivatives: energy–momentum

Multivector Representation. The base space $\mathbb{R}^5$ corresponds to the real variables set $\{ct, x, y, z, \kappa_0\alpha\} \leftrightarrow \{x^u; u = 0, 1, 2, 3, 4\}$ that is: time, 3-D space and action (in units of distance introducing the universal speed of light in vacuum c and the system under observation dependent $\kappa_0 = \lambda_{Compton}^{system\, with\, energy\, mc^2}/h = c/E\,[= 1/m_0c]$). In the most common approach to physics time is usually taken as an independent evolution coordinate and action (matter and interaction) is distributed in space, then we need to consider the functions $x(t), y(t), z(t)$ and $w(t, x, y, z) = \kappa_0\alpha(t, x, y, z)$. The linear forms for the description are the nested vectors

$$dS = \sum\nolimits_\mu dx^u e_u\, ;\ u = 0, 1, 2, 3, 4 \qquad 5-D$$
$$ds = \sum\nolimits_\mu dx^\mu \mathrm{e}_\mu\, ;\ \mu = 0, 1, 2, 3;\ \mathrm{e}_\mu = e_4 e_\mu \qquad 4-D$$
$$d\mathbf{x} = \sum\nolimits_i dx^i \mathbf{e}_i\, ;\ i = 1, 2, 3\ ;\ \mathbf{e}_i = \mathrm{e}_0\mathrm{e}_i \qquad 3-D$$

members of a Clifford algebra generated by the definition of a quadratic form

$$dS^2 \equiv (dS)^2 = \left(\sum\nolimits_\mu dx^u e_u\right)^2 = \sum\nolimits_{\mu\nu} g_{uv}^{START} dx^u dx^v,$$
$$g_{uv}^{START} = diag\,(1, -1, -1, -1, -1)\ ,\ e_u e_v = -e_v e_u,$$
$$e = e_0 e_1 e_2 e_3 e_4 = -e^\dagger \qquad e_u e = e e_u \text{ with } e^2 = 1.$$

The quadratic form which is more relevant for Physics considers that observable objects are extended in space and then an action density α in space-time is required.

Define $m(\mathbf{x}, t)c^2 = \varepsilon_{total}(\mathbf{x}, t)$, also the (Clifford algebra valued) volume inverse $e_0e_1e_2e_3/\triangle x\triangle y\triangle z\triangle t$

Table 1.

Quadratic form	Dim/diff. Op	Group
$l^2 = x^2 + y^2 + z^2 \qquad (\triangle t)$	3-D ∇, ∇^2	Galileo
$s^2 = (ct)^2 - (x^2 + y^2 + z^2)$	4-D $D, \Box^2$	Poincaré
$S^2 = (ct)^2 - \left(x^2 + y^2 + z^2\right) - w^2$	5-D K, K^2	Dynamics

Table 2.

$w = \kappa_{(0)}\overline{a}, \qquad \kappa_{(0)} = \dfrac{d_{(0)}}{h} = \dfrac{c}{E_{(0)}}$	$(\overline{a})^2 = \sum_\mu a_\mu^2$	START

Table 3.

$R^3 \overset{Q\,Form}{\Longrightarrow} Cl_3$ e_0 (observer) and e_4 fixed	*space where **given*** **observer** *describes the system*
$R^4 \overset{Q\,Form}{\Longrightarrow} Cl_{1,3}$ e_4 fixed (system is defined)	general *space-time* *to describe **given*** **system**
$R^5 \overset{Q\,Form}{\Longrightarrow} Cl_{1,4}$ ——————————————— $Q\,Form$ $\mathrm{dS}^2 = \mathrm{g}_{uv} dx^u dx^v$	general *space-time-action* *where **any*** **observer** *describes **any*** **system** as bundle of null trajectories

of the space-time volume, and the space-time Laplacian operator $\Box = \sum_\mu e^\mu \partial_\mu$ such that along $b = \sum_\mu b^\mu e_\mu$ the directional change operator is $db \cdot \Box = \sum_\mu db^\mu \partial_\mu$ (we apply four times for $b = cte_0, xe_1, ye_2, ze_3$). Then we can obtain the sum of the **directed** changes of the action density w:

$$\mathbf{a}(\mathbf{x},t)e_4 = \kappa_0 \alpha(\mathbf{x},t)e_4 = \kappa_0 \frac{m(\mathbf{x},t)c^2 \triangle t}{\triangle x \triangle y \triangle z \triangle t} e = \frac{1}{m_0 c} \frac{m(\mathbf{x},t)c^2 \triangle t}{\triangle x \triangle y \triangle z \triangle t} e,$$

$$\mathbf{a}(\mathbf{x},t)e_4 = \frac{(m(\mathbf{x},t)/m_0)\, c \triangle t}{\triangle x \triangle y \triangle z \triangle t} e = \frac{w(\mathbf{x},t)}{\triangle x \triangle y \triangle z \triangle t} e = w(\mathbf{x},t)e,$$

$$e\, dw = \sum\nolimits_\mu \left[(\partial_\mu w(\mathbf{x},t))\, dx^\mu\right] e_\mu e = \sum\nolimits_\mu \left[(p_\mu)\, dx^\mu\right] e_\mu e,$$

the contributions $[(p_\mu)\, dx^\mu]\, e_\mu$ to the action differential, using $[e_\mu e]^2 = -g_{\mu\mu}$, allow to write for the START squared differential

$$(dS)^2 = (dS)\,(dS)^\dagger = \left(1 - (\kappa_0 p_0)^2\right)(cdt)^2 + - \left(\left(1 - (\kappa_0 p_1)^2\right)(dx)^2 + \left(1 - (\kappa_0 p_2)^2\right)(dy)^2 + \left(1 - (\kappa_0 p_3)^2\right)(dz)^2\right)$$

"energy-momentum p_μ 'generates' curvature in $(dS)^2$ "

$$\mathbf{a}(\mathbf{x},t)e_4 = \kappa_0 \alpha(\mathbf{x},t)e_4 = \kappa_0 \frac{m(\mathbf{x},t)c^2 \triangle t}{\triangle x \triangle y \triangle z \triangle t} e = \frac{1}{m_0 c} \frac{m(\mathbf{x},t)c^2 \triangle t}{\triangle x \triangle y \triangle z \triangle t} e$$

$$\mathbf{a}(\mathbf{x},t)e_4 = \frac{(m(\mathbf{x},t)/m_0)\, c \triangle t}{\triangle x \triangle y \triangle z \triangle t} e = \frac{w(\mathbf{x},t)}{\triangle x \triangle y \triangle z \triangle t} e = w(\mathbf{x},t)e$$

$$e\, dw = \sum\nolimits_\mu \left[(\partial_\mu w(\mathbf{x},t))\, dx^\mu\right] e_\mu e = \sum\nolimits_\mu \left[(p_\mu)\, dx^\mu\right] e_\mu e$$

the contributions $[(p_\mu)\, dx^\mu]\, e_\mu$ to the action differential, using $[e_\mu e]^2 = -g^{(0)}_{\mu\mu}$, allow to write for the START squared differential (note $\kappa_0 p_\mu$ is adimensional)

$$(dS)^2 = (dS)\,(dS)^\dagger = \left(1 - (\kappa_0 p_0)^2\right)(cdt)^2 + - \left(\left(1 - (\kappa_0 p_1)^2\right)(dx)^2 + \left(1 - (\kappa_0 p_2)^2\right)(dy)^2 + \left(1 - (\kappa_0 p_3)^2\right)(dz)^2\right),$$

for variable p_μ we can say that **"energy-momentum $p_\mu(\mathbf{x},t)$ 'generates' curvature in a 4-D space-time subspace of $(dS)^2$ "**. In general

$$dS^2 = g^{uv}dx_u dx_v = g^{\mu\lambda}dx_\mu dx_\lambda$$
$$g^{\mu\lambda} = g^{\mu\lambda}_{(0)} - \vartheta^{\mu\lambda}\,;$$
$$u,v = 0,...,4\,;\ \mu,\lambda = 0,...,3 \quad (2)$$

presenting $(dS)^2$ as a (locally embedded) 4-D curved space. Here $p_\mu = \partial_\mu \alpha(\mathbf{x},t)$ is a momentum density and $w(\mathbf{x},t)$ a distance equivalent to a **reduced action** density.

2.2. Carriers of Energy and Momentum

A system to be studied, characterized by an action distribution, a scalar density at each point of space–time, where the total density of action is a sum of contributions which for description purposes can be called the constitutional action corresponding to the carriers mass and the interaction part of the action corresponding to a description dependent part.

This paper analyzes gravity, then it centers on the system-environment interaction part. In that case the system under the gravitational interaction is characterized by a mass distribution $m(x,t)$. A particular case of interest is where the mass distribution can be written as $m_0\rho(x,t)$ which, because of its properties, can be called an elementary carrier, the density $\rho(x,t) \geq 0$, then it requires to be the real square of an analytic function $\rho(x,t) = |\psi\dagger(x,t)\psi(x,t)|$ (see Keller and Weinberger 2003), the study of this relation being in the foundations of quantum mechanics. Otherwise the (mass) density is only the time-like component of a space–time vector $J(x,t) = \{\rho, j_1, j_2, j_3\}$ where the space-like components are the currents $j_i(x,t)$, the analysis below uses this covariant current.

A fundamental quantity related to the action density a is the energy–momentum density $E-\mathbf{P}$ and its scalar analytical action function S: (here $X = x_\mu e^\mu$, $e^\mu e^\nu = g^{\mu\nu}$ and $\Box = (\partial/\partial x^\mu)\, e^\mu$)

$$\Box a = \frac{1}{c}Ee_0 + p_1e^1 + p_2e^2 + p_3e^3, \quad (3)$$
$$S \ = X \cdot \Box a = Et - x^1p_1 - x^2p_2 - x^3p_3 \quad (4)$$

(S is the function used by Schrödinger or de Broglie). For that part of E corresponding to the internal system's energy we have the definition

$$E(x,t) = m(x,t)c^2.$$

2.3. A Covariant Extension of Newtonian Gravitation

Let us formally show that the gravitation (covariant in space-time) equations in their standard textbook form are analytical properties of the third derivatives of the action density attributed to a test carrier as induced by a collection of interacting carriers (carriers for which the action and the energy–momentum is shared but partitioned).

The energy per carrier can be considered the derivative of a scalar field. For this type of carriers the shared (vector) momentum, with a non null rotational part, can not be considered the gradient of a scalar field. What we are doing in this particular case is to assume that we describe a set of carriers as interacting by partitioning the energy among them, allowing the partitioning of the momentum, which is a quantity with a sign related to directions in space, to be the sum of the momenta of the different carriers. That is the sum could be any number including zero whereas the individual momenta will be described as the result of adding for each the overall momentum plus the momentum induced by the interactions among the carriers.

Then if in the reference frame of a given observer the induced action density, denoted by $a_e(X)$, per unit "charge" where in this case the charge responsible for the interaction is the mass (⇒puc) of a test carrier at space–time point $X = x^\mu e_\mu$ (here $\mu = 0,1,2,3$ and $x^0 = ct$ and the space vectors $\mathbf{q} = q^i e_i = q_i e^i$, $i = 1,2,3$ are written in bold face letters), then the related energy density (puc) and the induced momentum density, **per unit charge of the test carrier**, would be

$$\mathfrak{E}_e(X) = \frac{\partial a_e(X)}{\partial t},$$
$$\mathbf{p}_e = p_{e,i}e^i = \left(\frac{\partial a_e(X)}{\partial x^i} + \Delta_R\, p_{e,i}\right)e^i, \quad (5)$$

and the, by definition, gravitational field intensity $\mathbf{E}$ is the force (puc) corresponding to this terms

$$\mathbf{E} = \left(\frac{\partial \mathfrak{E}_e(X)}{\partial x^i} + \frac{\partial p_{e,i}}{\partial t}\right)e^i = \nabla\mathfrak{E}_e(X) + \frac{\partial \mathbf{p}_e}{\partial t},$$

with time dependence

$$\frac{\partial \mathbf{E}}{\partial t} = (\frac{\partial^2 \mathfrak{E}_e(X)}{\partial t\partial x^i} + \frac{\partial^2 p_{e,i}}{\partial t \partial t})e^i$$
$$= \frac{\partial^3 a_e(X)}{\partial t \partial x^i \partial t}e^i + \frac{\partial^2\left(\Delta_R p_{e,i}\right)}{(\partial t)^2}e^i.$$

Otherwise, by definition of interacting carriers, we have added in (5) the term $\Delta_R\, p_{e,i}e^i$ as the effect of the conservation of **interaction transverse momenta** between the field representing the rest of the carriers with that sort of charges. In START this is, by definition, the origin of a geomagnetic field intensity $\mathbf{B} = B_k e^k$ that will appear as the curl of the momentum (puc) of an interaction field acting on a carrier, the space axial vector

$$\mathbf{B} = \left(\frac{\partial p_{e,i}(X)}{\partial x^j}\right)e^j \times e^i = \nabla \times \mathbf{p}_e,$$

with time dependence

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial^2 p_i(X)}{\partial t \partial x^j} e^j \times e^i.$$

The space variation of $\mathbf{E}$, including for this carriers the **interaction transverse momenta**,

$$\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + \nabla \times \mathbf{E}, \tag{6}$$

will then also include a transversal (rotational) term

$$\nabla \times \mathbf{E} = \frac{\partial^2 p_j(X)}{\partial x^i \partial t} e^i \times e^j = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{(2nd Grav Equation)}$$

relation which is the direct derivation in START of the gravitational equivalent of a well known Maxwell equation. The scalar term $\nabla \cdot \mathbf{E}$ being a divergency of a vector field should be defined to be proportional to a source density

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_g}\rho = \sum_i \left(\frac{\partial^3 a_e(X)}{\partial x^i \partial x^i \partial t} \right) = \frac{\partial}{\partial t} \nabla^2 a_e(X), \quad \text{(1st Grav Equation)}$$

and will be given full physical meaning below.

For the space variation of $\mathbf{B}$ we have

$$\nabla \mathbf{B} = \nabla \cdot \mathbf{B} + \nabla \times \mathbf{B}.$$

The first term vanishes identically in our theory because it corresponds to the divergence of the curl of a vector field

$$\nabla \cdot \mathbf{B} = 0, \quad \text{(3rd Grav Equation)}$$

while the last term, using $U \times V \times W = V(U \cdot W) - (U \cdot V)W$

$$\nabla \times \mathbf{B} = \nabla \left(\nabla^2 a_e(X) \right) - \nabla^2 \mathbf{p}_e = \mu_g \left(\mathbf{J} + \epsilon_g \frac{\partial \mathbf{E}}{\partial t} \right), \quad \text{(4th Grav Equation)}$$

where the additional dimensional constant μ_g is needed to transform from time units (used in the definition of a current $\mathbf{J} = \nabla \left(\nabla^2 a_e(X) \right) / \mu_g$) into distance units and $\epsilon_g \mu_g$ will have then units of T^2/D^2 or inverse velocity squared, in fact (see below) c^{-2}. The 4th Grav Equation, defining $\mathbf{J}$, is related to the analogous of the 1st Grav Equation and to the analogous of the 2nd Grav Equation, also to a Lorentz transformation of the 1st Grav Equation.

A comprehensive theory of gravitation should use this formal structure, by construction is a one-to-one map of the electromagnetic Maxwell equations and effects. This is a universal interaction structure, for each type d of charge there is a pair (ϵ_d, μ_d) of coupling constants.

The derived gravitational interaction equations extend and are formally equivalent to the original Newton equations, then they are: first local equations and second linear in the sources (ρ and $\mathbf{J}$).

Both the 4th Grav Equation, defining $\mathbf{J}$, related to a Lorentz transformation of the 1st Grav Equation, defining ρ,can immediately be integrated. The space divergence of a non-solenoidal vector field like $\mathbf{E}$ is immediately interpreted as its 'source' given that

$$\Delta \mathbf{E} = (\nabla \cdot \mathbf{E})\, \mathfrak{S} \Delta \mathbf{x},$$

and this equation is integrated using the standard geometric theorem that the volume integral of a divergency $\nabla \cdot \mathbf{E}$ equals the surface integral of the normal (to the surface) component of the corresponding vector field $\mathbf{n} \cdot \mathbf{E}$. Where $\mathbf{n}$ is a unit vector perpendicular to the surface S (in the text-book formula below $\mathfrak{S} = 4\pi r^2$ corresponding to an integration sphere of radius r containing a spherically symmetric source density $\rho(r)$ generating a force field per unit charge $\mathbf{E} = E(r)\frac{\mathbf{r}}{r}$) of the integration volume $V = 4\pi r^3/3$:

$$\int_V (\nabla \cdot \mathbf{E})\, dV = \int_V \frac{4\pi}{\epsilon_g} m(r') r'^2 dr' = \frac{1}{\epsilon_0} M$$
$$= \int_S E(r) \frac{(\mathbf{r} \cdot \mathbf{n})}{r} d\mathfrak{S} = 4\pi r^2 E(r)$$
$$\mathbf{E} = E(r)\frac{\mathbf{r}}{r} = \frac{M}{4\pi \epsilon_g r^2} \frac{\mathbf{r}}{r} = \frac{GM}{r^2}\frac{\mathbf{r}}{r}, \qquad \mathbf{r} \cdot \mathbf{n} = r.$$

That is: *the inverse square law of the Newtonian and Coulombic forces are geometrical consequences of the definition of interaction among "charged" carriers.* But this is not a derivation of the values of the *Newtonian and Coulombic* constants G and ϵ_0.

In the case of the, generated by a current, gravomagnetic force field intensity $\mathbf{F}$, being a space bivector, it is also a direct geometrical consequence that its source can (must) be a current vector density $\mathbf{J}$. For a small ($l \ll r$) current source at the origin of coordinates:

$$\int_V (\nabla \times \mathbf{F})\, dV = \int_S F(r) \left(\mathbf{r}^t(\theta, \phi) \times \mathbf{n} \right) d\mathfrak{S}$$
$$= 4\pi r^2 f F(r) \mathbf{r}^{ct}, \tag{7}$$

$$\mathbf{r}^t(\theta, \phi) \cdot \mathbf{r}^{ct} = 0, \tag{8}$$

$$\int_V 4\pi \mu_g \mathbf{J} \delta(r') r'^2 dr'$$
$$= \mu_g M \mathbf{r}^{ct} = 4\pi r^2 f F(r) \mathbf{r}^{ct} \Longrightarrow \tag{9}$$

$$\Longrightarrow \mathbf{F} = F(r) \mathbf{r}^t(\theta, \phi) = \frac{\mu_g}{4\pi r^2} \frac{M}{f} \frac{\mathbf{r}^t(\theta, \phi)}{r},$$
$$\left(\mathbf{r}^t\right)^2 = \left(\mathbf{r}^{ct}\right)^2 = 1 \tag{10}$$

and its *Amperian inverse square law is also a geometrical consequence of the definition of interaction among massive carriers*. The mass, obtained from the derivative of the action density , is a pseudo-scalar in START, then the square of the field intensities will be negative.

2.4. Tautology between Fields and Charges

Consider the particular case of an initial situation without gravitational phenomena being present $\mathbf{E}_g = 0$, $\mathbf{J}_g = 0$, $\mathbf{B} = 0$ and $m(x,t) = 0$, and that in the process of creating a pair of 'interacting carriers' with charges M, an initial pulse of current $\mathbf{J}_g(\mathbf{r},t) = M\mathbf{v}(\mathbf{r})\delta(t_0)$ is assumed to have been generated, this induces a gravitational field for $t > t_0$ from the Maxwell Equations derived above:

$$\begin{aligned}\partial_t \mathbf{E}_g &= -\mathbf{J}_g(\mathbf{r},t)/\epsilon_g + \nabla\times\mathbf{B}_g/\epsilon_g\mu_g \\ &= M\mathbf{v}(\mathbf{r})\delta(t_0)/\epsilon_g, \quad (11)\\ \mathbf{E}_g(\mathbf{r},t) &= -M\mathbf{v}(\mathbf{r})/\epsilon_g,\end{aligned}$$

where $m(\mathbf{r},t) = -\nabla\cdot\mathbf{v}(\mathbf{r})$ and

$$\partial_t \mathbf{B}_g(\mathbf{r},t) = -\nabla\times\mathbf{E}_g(\mathbf{r},t) = M\nabla\times\mathbf{v}(\mathbf{r})/\epsilon_g \quad (12)$$

showing that this virtual mechanism (in our process to establish a partitioning of energy and momentum) can give, to the collection of created carriers, physical properties as far as the divergence of the current pulse creates a charge and the rotational of the current pulse of velocity field $\mathbf{v}(\mathbf{r})$ a gravitational wave.

If in the process of creating a carriers-pair the vector current pulse is a decaying pulse, (inversely with distance square, that is a concentric spheres perturbation of the energy distribution $\partial_t a(\mathbf{r})$ with an equal radial total strength) :

$$\mathbf{v}(\mathbf{r}) = \frac{-1}{4\pi r^2}\frac{\mathbf{r}}{r}, \qquad 4\pi r^2\mathbf{v}(\mathbf{r}) = \frac{\mathbf{r}}{r},$$

then the generated **permanent** gravitational field is

$$\mathbf{E}(\mathbf{r}) = \frac{\partial^2 a_e(\mathbf{r})}{\partial r\partial t} = \frac{M}{\epsilon_g 4\pi r^2}\frac{\mathbf{r}}{r}$$

representing the creation of the gravitational effect of a permanent mass M. The linearity of the equations allows the consideration of a mass distribution $m(x,t)$. We clearly see the inseparability of the concept of charge and of the field of that source charge. The concept of action density in space–time is fundamental in our discussion.

2.5. The Energy of the Gravitational Field

What is the gravitostatic energy stored in a collection of N static point masses m_i located at position vectors $\mathbf{r}_i$ (where i runs from 1 to N). In particular: how much work would we have to do in order to assemble the masses, starting from an initial state in which they are all at rest and very widely separated?

By construction the static gravitational field is conservative, and written in terms of a scalar potential: $\mathbf{E} = -\nabla\phi$. The gravitational force on a mass m is written $\mathbf{f} = m\mathbf{E}$. The work *we* would have to do against gravitational forces in order to move the mass from point P to point P' is $W = -\int_P^{P'}\mathbf{f}\cdot d\mathbf{l}$. No work is needed to place the first mass in position at $\mathbf{r}_1$. In order to bring the second mass into position at $\mathbf{r}_2$, we have to do work $W_2 = -\frac{1}{4\pi\epsilon_{g0}}\frac{m_2 m_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$ against the gravitational field generated by the first mass. This result can easily be generalized to N masses: $W = -\frac{1}{4\pi\epsilon_{g0}}\sum_{i=1}^{N}\sum_{j>i}^{N}\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}$. Alternatively, this is the kinetic energy which would be released if the collection were dissolved, and the masses returned to infinity. Our approach is a field point of view, where is this potential energy stored? For this consider the potential energy of a continuous mass distribution. Write

$$\begin{aligned}W &= \frac{1}{2}\int \rho\phi d^3\mathbf{r},\\ \phi(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_{g0}}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}',\end{aligned}$$

where $\phi(\mathbf{r})$ is the scalar potential generated by a continuous mass distribution. Substitute $\rho = \epsilon_{g0}\nabla\cdot\mathbf{E}$ to write

$$W = \frac{\epsilon_{g0}}{2}\int\phi\nabla\cdot\mathbf{E}d^3\mathbf{r}.$$
$$\nabla\cdot(\mathbf{E}\phi) = \phi\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla\phi. \text{ and } \nabla\phi = -\mathbf{E},$$

to obtain

$$W = -\frac{\epsilon_{g0}}{2}\left[\int\nabla\cdot(\mathbf{E}\phi)\,d^3\mathbf{r} - \int\mathbf{E}^2 d^3\mathbf{r}\right]$$
$$W = -\frac{\epsilon_{g0}}{2}\left(\int_S \phi\mathbf{E}\cdot d\mathbf{S} + \int_V \mathbf{E}^2 dV\right),$$

here V is a large volume which encloses all of the masses, and $\mathbf{S}$ is its bounding surface where $\phi\mathbf{E}$ decreases as $1/r^3$. Then W reduces to

$$W = -\frac{\epsilon_{g0}}{2}\int\mathbf{E}_g^2 d^3\mathbf{r},$$

where the integral is over all space. The (negative) potential energy of a continuous mass distribution is stored in the gravitational field. By construction we then find that a gravitational field possesses an *energy density*

$$U = -\frac{\epsilon_{g0}}{2}\mathbf{E}_g^2.$$

The analysis above explicitly avoided the problem of the self-gravitational energy. Otherwise we have to add $\sum_i W_i$ the energy required to have the collection of masses. To write

$$W = -\frac{\epsilon_{g0}}{2}\int E^2 d^3\mathbf{r} = \frac{1}{2}\sum_{i=1}^{N} m_i\phi_i + \sum_{i=1}^{N} W_i,$$

The idea of locating gravitational potential energy in the gravitational field is inconsistent with the existence of point masses. One way out of this difficulty is to admit that all elementary masses, such as electrons, are not points, but distributions of mass.

We can find the interaction energy for two overlapping gravitational fields $\mathbf{E}_P$ and $\mathbf{E}_e$, avoiding complicated scalar products of vectors and explicit 3-dimensional integrals, using a model of equal concentric masses: $m_P = M$ and $m_e = m$ with spherical shell distributions with radius $b \ll r_B$ and r_B respectively, then the self-gravitational parts and the interaction energy

$$\begin{aligned} W_P &= -\frac{GM^2}{2}\int_b^\infty \frac{1}{r^2}dr = -\frac{GM^2}{2b} \text{ and} \\ W_e &= -\frac{Gm^2}{2}\int_{r_B}^\infty \frac{1}{r^2}dr = -\frac{Gm^2}{2r_B}, \end{aligned} \quad (13)$$

$$\begin{aligned} W_{int} &= \frac{-1}{8\pi G}\int_b^\infty (\mathbf{E}_P + \mathbf{E}_e)^2 4\pi r^2 dr - W_P - W_e \\ &= -GMm\int_{r^B}^\infty \frac{1}{r^2}dr = -\frac{GMm}{r_B} \end{aligned} \quad (14)$$

again equivalent to the product of the masses divided by the distance between the masses. Reminder $(\mathbf{E}_a)^2 = -|\mathbf{E}_a|^2$ because in START masses are 5-D pseudo-scalars.

2.6. The Action of a Body in a Gravitational Field

For the massive carrier field at rest

$$\mathcal{E}^2 = (\mathcal{E}_0 + \Delta\mathcal{E})^2,$$

where $\Delta\mathcal{E}$ is a gauge-free energy contribution and $\mathcal{E}_0 = m_0c^2$.

The concept of test carrier in gravitation, even in general relativity for example in the Schwarzschild solution, is compatible with the Newtonian limit for the interaction gravitational energy

$$\Delta\mathcal{E}(r) = -m_0\frac{GM}{r},$$

where M is the mass of 'the source' we are exploring with the test particle. Consider the changes in energy-momentum arising from the interaction:

$$\begin{aligned} \mathcal{E}^2 - \mathcal{E}_0^2 &= \mathcal{E}_0^2 + 2\mathcal{E}_0\Delta\mathcal{E} + (\Delta\mathcal{E})^2 - \mathcal{E}_0^2 \\ &= (pc)^2 - (p_0c)^2 = 2\mathcal{E}_0\Delta\mathcal{E} + (\Delta\mathcal{E})^2 \to \\ &\qquad - 2m_0c^2m_0\frac{GM}{r} + \left(m_0\frac{GM}{r}\right)^2, \end{aligned}$$

$$\begin{aligned} p^2 &= (m_0v + \triangle p)^2 \\ &= \left((m_0v)^2 - 2m_0^2\frac{v}{c}\frac{GM}{cr} + \left(m_0\frac{GM}{cr}\right)^2\right) \end{aligned}$$

and space spherical coordinates t, r, θ, ϕ, we obtain for the action $(da)^2 =$

$$\begin{aligned} &\left((m_0c^2)^2 - 2m_0c^2\frac{m_0GM}{r} + \left(m_0\frac{GM}{r}\right)^2\right)(dt)^2 \\ &- \left((m_0v)^2 - 2m_0^2\frac{v}{c}\frac{GM}{cr} + \left(m_0\frac{GM}{cr}\right)^2\right) \\ &\times\left\{(dr)^2 + r^2\left[(d\theta)^2 + \sin^2\theta\,(d\phi)^2\right]\right\} \end{aligned} \quad (15)$$

which, on multiplication by the 'action as a distance' factor $\kappa_0 = 1/m_0c$, contains, in the limit of $r \gg GM/c^2$, the same physical implications as the Schwarzschild metric but with the speed of light always c for all observers.

The negative of the second derivative of a field is proportional to the source, by definition. The source of the "curvature" is either the mass density (in the second term) or the energy of the gravitational field density (in the last term). The energy of the gravitational field, from the second derivative of $(m_0GM/(cr))^2$, turns out to be negative (Dark Energy like) as corresponds to an attractive interaction.

The square root of (15) divided by dt is the ordinary Lagrangian for a particle in a gravitational field.

Acknowledgment

The technical assistance of Mrs. Irma Vigil de Aragón are gratefully acknowledged. This project was additionally supported by the UNAM program PAPIIT number IN113102-3.

Manuscript received March 5, 2006

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