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Electromagnetic Fields: Their Classification in General Relativity and Propagation in a Vacuum

Contents

Abstract

A practical and concise classification of electromagnetic fields in general and special relativity is formulated. We give a simple scheme of determination of reference frame co-moving with electromagnetic field which is applicable to all types of fields with the exception of pure null one in which such a frame cannot be introduced in principle, since only this case corresponds to propagation of field with the velocity of light in a vacuum. The co-moving frame is introduced in the energy sense: the Poynting vector in this frame vanishes. As an important example, the Liénard–Wiechert field is considered, and it is explicitly shown that it propagates with a sub-luminal velocity. The mathematical technique is provided in detail.

1. Introduction

It seems that everybody knows what is the electromagnetic field. Let us however force ourselves to look at this field more critically and without the common prejudices. In fact, this needs making a certain effort. Thus, what is the electromagnetic field, after all? Let us primarily have in mind the classical (*i.e.*, non-quantum-theoretical) physics, in 3+1 dimensions indeed. And, nevertheless, it will be not so bad to have an idea what should represent, in the main, the specific features of the quantum physics in these respects. Below we first mention some features, aspects and even nuances of the general approach we propose to the reader, as well as those we are going to specifically include in this communication, and then we describe some individual topics in more detail. There will be also found some results of application of these ideas, and interpretation of those results.

So, in this (classical) case, what does the difference, say, between the special relativistic and general relativistic approaches, consist of? And what is common between them? In particular, gravitation does automatically regularize the global characteristics, so that in general relativity (at least, some) local singularities do not anymore lead to an integral energy (or mass) divergence,— recall in this respect the well-known "eternal" solutions of Schwarzschild, Reissner and Nordström, and quite a few of others. Thus one is not compelled to introduce to this end the more or less artificial non-linearities, for example, in the electromagnetic field (like those of Born and Infeld — and also Schrödinger — or, in the quantum theoretical case, of Schwinger). But this is one of the few really important differences between special and general relativistic approaches (probably, resulting from the much more self-consistent nature of the latter approach). Usually, these two approaches differ one from another merely in absence or presence of the gravitational field (curvature) in our description of Nature. These and other conclusions on such subjects can have an important effect upon molding our opinion of what in the electromagnetic field is the most specific and most profound.

The Maxwell field is described with full completeness in special relativity, see, for example, [1,11,14]. Its characteristic features are that this field is linear (its Lagrangian density is quadratic in the field tensor), and its potential is a vector (concerning the variational principle, it is better to say here: covector). Since in the most fundamental cases in physics the Lagrangian densities contain the simplest possible invariants describing systems under consideration (best examples: the above-mentioned Lagrangian of the free electromagnetic field and the gravitational field Lagrangian containing the simplest possible invariant of the metric tensor and its derivative to the extent sufficient to obtain non-trivial partial differential equations), the Lagrangian describing the interaction between electromagnetism and a distribution of electric charge in its motion is also well determined as being proportional to the scalar product of the four vector of the current density and the electromagnetic four potential; remember here the above-mentioned "3+1".

Then we have, merely as corollaries, the following important properties of the Maxwell field (both in special and general relativity): The field tensor, naturally, is skew-symmetric and has rank two. The electric and magnetic parts of this field are interchanged while we apply the dual conjugation to the field tensor (with one change of the sign due to the spacetime signature, so that the repeated dual conjugation results in the *minus* the same field). The stress-energy (energy-momentum) tensor is invariant under substitution of the dually conjugated field tensor (instead of the original one), and has its trace identically equal to zero. And so on, including the standard expressions of the electromagnetic energy density, the Poynting vector, the 3×3-stress. The 3+1-decomposition is general covariant and occurs similarly in special and general relativity; this is important not only in the just given remarks, but even more in the very description of the electric and magnetic parts of the field, see [10].

But what will occur when we change any of our initial suppositions (there are very few of them)? For example, is it possible to change the 3+1 dimensionality without affecting the corollaries? — In general, no; more strictly speaking, such a change is possible only to an odd number of spatial dimensions (the temporal one always remains to be 1). Thus, for example, in 2+1 and 4+1 dimensions there cannot exist an electromagnetic field endowed with the same properties as in 3+1. Moreover, in 2+1 any field constructed from the vector (rank 1) potential, even if we accept also non-linear field equations, essentially has properties of a fluid and not of electromagnetic field, and inhomogeneity of the "field" equations (the term usually written in the right-hand side, like that with j^μ in Maxwell's equations) is easily and unequivocally shown to represent rotation of fluid, and not a source of the field (there are more fine points, but they only aggravate the situation and not reconcile the contradictions with the naïvely expected electromagnetic interpretation of the 2+1-field). See more comments (in particular, as Table I) in [8], among them on the necessity to change from 1-form as the electromagnetic potential in 3+1 to 2-form in 5+1. Another problem, described in the same paper, is that any electromagnetic field, including the static and stationary ones, cannot have a non-relativistic approximation, thus being of an intrinsically relativistic nature, and the principle of equivalence of inertial and gravitating masses should be correspondingly generalized in its application to all such fields; see also [3,4,6,7].

In this communication we use only one concrete example of electromagnetic fields, the Liénard–Wiechert one, but this obviously does not restrict our considerations to special relativity.

2. Classification of electromagnetic fields

The classification of electromagnetic fields is based on existence of only two invariants built with the field tensor $F_{\mu\nu}$, while all other invariants are merely algebraic functions of these two (if not vanish identically). The first invariant is $I_1 = F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2)$, and the second, $I_2 = F^{\,*}_{\mu\nu}\, F^{\mu\nu} = 4\mathbf{E} \bullet \mathbf{B}$, see (B.9),(B.2) and (B.3); the definition of I_2 contains dual conjugation of $F_{\mu\nu}$,

$$\begin{aligned} F^{\,*}_{\mu\nu} &:= \frac{\sqrt{-g}}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \\ F^{\mu\nu}_{\,*} &:= -\frac{1}{2\sqrt{-g}}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}. \end{aligned} \tag{1}$$

Here $\epsilon_{\mu\nu\alpha\beta}$ is the completely skew-symmetric object (not exactly a tensor) with $\epsilon_{0123} = +1$, known as the Levi-Cività symbol. In fact, only the squared I_2

is really invariant, and I_2 itself is a pseudo-invariant (axial scalar) which acquires the factor $\mathrm{sign}(J) = J/|J|$ by a general transformation of coordinates, J being the Jacobian of the transformation, so that the concrete sign of I_2 should not matter in our classification.

In terms of I_1 the invariant classification suggests three types of fields: $I_1 < 0$ is the electric type (the electric field dominates), $I_1 > 0$ gives the magnetic type, and to $I_1 = 0$, the null type corresponds. The pseudo-invariant I_2 permits to work out the classification in more detail: we get additional subtypes, impure ($I_2 \neq 0$) and pure ($I_2 = 0$).

It is important that the pure electric case permits to completely eliminate the magnetic field, and similarly, the pure magnetic field permits to completely eliminate the electric field; these eliminations are realized when we elect specific reference frames which can be called frames co-moving with the respective electromagnetic fields. However, the pure null electromagnetic field in a vacuum permits only to find reference frames in which the electric and magnetic field intensities would take any desired finite (nonzero and non-infinite) and equal values, while, of course, the field will continue to pertain to the same pure null type (both fields $\mathbf{E}$ and $\mathbf{B}$ will be ever equal in their absolute values and mutually orthogonal, according to the structure of both invariants). This last property is closely related to the Doppler effect (not only in the sense of the frequency, but — and more profoundly — also of the field intensity). In particular, a complete elimination of the pure null type field is 'possible' only asymptotically (in less rich-in-content terms, this means 'impossible'), since reference frames moving with the speed of light with respect to any other reference frame are degenerate, thus simultaneous elimination of both electric and magnetic fields (in the case of the pure null type) is impossible in any admissible (non-degenerate) frame.

The impure electric, magnetic, and null types obviously do not permit such manipulations with the three-dimensional parts $\mathbf{E}$ and $\mathbf{B}$ of the electromagnetic field (in the impure electric and magnetic cases it is impossible to transform away the counterparts of these respective fields). However, it is then possible to find frame(s) in which these vectors are mutually parallel, so that the Poynting vector vanishes (this was previously attained by making one of them equal to zero). In both cases, pure and impure ones, this means that the corresponding frame is co-moving with respect to the electromagnetic field. Thus the only exclusion dictated by the relativistic causality law, is that of the pure null type which includes the plane-waves-in-vacuum solutions.

We see how powerful is this simple classification of electromagnetic fields working both in special and general relativity. The especially characteristic reference frames described above, may be called canonical frames related to the concrete types of electromagnetic fields. It is obvious that these frames in general are non-inertial; only in very special cases they may become inertial, though their belonging to special relativity is then merely necessary, but not sufficient.

3. The Liénard–Wiechert solution

Let us now briefly consider the famous Liénard–Wiechert (below abbreviated as LW) solution pertaining to special relativity, but exceptionally rich of physical content including its relevance to specific non-inertial reference frames. It was discovered more than one hundred years ago by A. Liénard [2] and E. Wiechert [17] as an exact solution of Maxwell's equations describing electromagnetic field of a pointlike electric charge in an arbitrary motion. A frequently used treatment of this solution can be found in [1], and its more general deduction, including the use of an arbitrary mixture of retarded and advanced potentials, in [14].

A simple and direct deduction of the LW solution (see for details [5, 9]) involves the use of the light cone concept, in this context, in fact a supposition of lightlike propagation of *information* from this pointlike source. We introduce a vector connecting the four-points (events) P' and P (P is the point of determination of the LW electromagnetic field, and P' the retarded point on the world line of a point-like electric charge, the LW field source),

$$R^\mu = x^\mu - x'^\mu(t'). \tag{1}$$

Of course, this is not a vector under more general transformations than the Lorentz ones (like the Euclidean 'radius vector' is a vector only in Cartesian systems). Let R^μ lie on the light cone (with its vertex either in P or in P'), thus this vector is null:

$$R^\mu R_\mu = 0. \tag{2}$$

Its projection onto u' is denoted as D, and onto the retarded three-space, as $\mathbf{D}^\mu$:

$$\begin{aligned} &D := u'^\mu R_\mu \equiv u' \cdot R, \\ &\mathbf{D}^\mu = R^\nu b^\mu_\nu = R^\mu - D u'^\mu, \quad \mathbf{D} \perp u'. \end{aligned} \tag{3}$$

The projector onto the three subspace orthogonal to u' is defined here as

$$b_{\mu\nu} := g_{\mu\nu} - u'_\mu u'_\nu \tag{4}$$

(*cf.* (A.2)); this and the null property (2) yield

$$\mathbf{D}^\mu \mathbf{D}_\mu = -D^2, \quad D = \sqrt{-\mathbf{D}^\mu \mathbf{D}_\mu}. \tag{5}$$

Let us call $\mathbf{D}^\mu$ the 'retarded spatially projected vector between P' and P.' Similarly, D is interpreted as the retarded three-dimensional distance between P' and P.

Thus one comes to the LW four potential

$$A^\mu = \frac{Qu'^\mu}{D} \tag{6}$$

and to the corresponding expression of $F_{\mu\nu}$:

$$F_{\mu\nu} = \frac{Q}{D^2}\left[R_\mu\left(a'_\nu + u'_\nu\frac{1-a'\cdot R}{D}\right)\right.$$
$$\left.-R_\nu\left(a'_\mu + u'_\mu\frac{1-a'\cdot R}{D}\right)\right]. \tag{7}$$

This is a specific type of skew-symmetric tensor sometimes called *simple bivector* since it represents an antisymmetrization of only two vectors, R^μ (1) and $U^\mu = \frac{Q}{D^2}\left(a'^\mu + u'^\mu\frac{1-a'\cdot R}{D}\right)$:

$$F_{\mu\nu} = R_\mu U_\nu - U_\mu R_\nu \tag{8}$$

which can be written as a 2-form $F = R\wedge U$, $R = R_\mu dx^\mu$ and $U = U_\mu dx^\mu$. Since $I_2 = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \equiv 0$ for any simple bivector (8), even with *arbitrary* R and U, the field is pure. Then it is pure electric since

$$I_1 = -\frac{2Q^2}{D^4} < 0 \tag{9}$$

(remarkably, the structure of I_1 is exactly Coulombian). Thus its magnetic part can be transformed away when one passes to a certain non-inertial reference frame.

4. Propagation of electromagnetic fields

It is well known that in a vacuum plane electromagnetic waves (as well as the electromagnetic field discontinuities) propagate with the fundamental velocity c. However, a mixture of non-radiative and radiative electromagnetic fields has another propagation velocity ($< c$). This occurs because of non-linearity (quadratic structure) of such characteristics of the electromagnetic field as its energy density and energy flux density (the Poynting vector). For these reasons, when we speak about 'propagation of information,' we do not speak strictly about propagation of electromagnetic field in the general sense: the question about the carrier of information in this concrete case (and even about this carrier's very existence) still remains open.

In a reference frame which is co-moving with electromagnetic field, the Poynting vector should vanish. This can occur for two alternative reasons (to be realized in this frame): either electric and magnetic vectors are mutually parallel (this is the impure classification subcase), or one of them is equal to zero (the pure subcase). The first case was considered by Wheeler [16] toward other ends. The second case pertains naturally to the LW field since this is a pure electric one (thus Wheeler's approach is not applicable, and the magnetic part can be transformed away *via* a proper choice of the reference frame). In fact, this possibility is scarcely encountered in literature (I even don't know any references), and it would be interesting to investigate it in more detail. We shall see that this task is much simpler than one could expect.

Remember the general form of the LW field tensor, (8): $F_{\mu\nu} = R_\mu U_\nu - U_\mu R_\nu$. Let us (algebraically) regauge the vector $U \rightarrow V = U + kR$ where k is a scalar function. This does not change the field tensor,

$$F_{\mu\nu} = R_\mu V_\nu - V_\mu R_\nu. \tag{10}$$

Applying now the 1-form definition of the magnetic vector in a τ-frame [10] and taking the monad as $\tau = NV$ where the scalar normalization factor is $N = (V\cdot V)^{-1/2}$, we obviously come to $\mathbf{B}= 0$ in this frame. The problem is thus reduced to a proper choice of k such that V will be a suitable real timelike vector with $V\cdot V > 0$. This method should work in our case (for a pure magnetic field, a similar technique can be applied, though requiring automatic representation of $*F$ as a simple bivector).

We see that

$$V^\mu = \frac{Q}{D^2}\left(a'^\mu + \frac{1-a'\cdot R}{D}\,u'^\mu + kR^\mu\right), \tag{11}$$

thus it was natural to include before k the scalar coefficient Q/D^2. Then

$$V\cdot V = \left(\frac{Q}{D^2}\right)^2\left[a'\cdot a' + \frac{(1-a'\cdot R)^2}{D^2} + 2k\right]. \tag{12}$$

In fact, k still remains arbitrary. Let it be

$$k = \frac{1}{2}\left[\frac{1}{D^2} - a'\cdot a' - \frac{(1-a'\cdot R)^2}{D^2}\right] \tag{13}$$

(the first term in the square brackets, $1/D^2$, got its denominator to fit the dimensional considerations). Finally,

$$V\cdot V = \left(\frac{Q}{D^3}\right)^2 > 0 \tag{14}$$

and

$$\hat{\tau}^\mu = Da'^\mu + (1-a'\cdot R)\,u'^\mu$$
$$+\frac{1}{2D}\left[1 - D^2a'\cdot a' - (1-a'\cdot R)^2\right]R^\mu \tag{15}$$

(it is clear that $\hat{\tau}\cdot\hat{\tau} = +1$). By its definition, the monad $\hat{\tau}$ describes the reference frame co-moving with the LW electromagnetic field: in this frame the Poynting vector of the field vanishes, and the electromagnetic energy flux ceases to exist due to the absence of magnetic part $\hat{\mathbf{B}}$ of the field in this frame (applicable at any finite distance D, not asymptotically). Really, (10) now can be rewritten as

$$F_{\mu\nu} = \frac{Q}{D^3}\left(R_\mu \hat{\tau}_\nu - \hat{\tau}_\mu R_\nu\right),$$

thus the expression of $\hat{\mathbf{B}}$ contains $\hat{\tau} \wedge R \wedge \hat{\tau} \equiv 0$. (See for definitions of the electric and magnetic vectors $\mathbf{E}$ and $\mathbf{B}$ [10].) Let us now calculate the electric vector $\hat{\mathbf{E}}$ in the frame $\hat{\tau}$. A combination of (15), (11), and (10) gives

$$F = R \wedge V = \frac{Q}{D^3} R \wedge \hat{\tau}. \qquad (16)$$

Then

$$\hat{\mathbf{E}} = *(\hat{\tau} \wedge *F) = \frac{Q}{D^3} * [\hat{\tau} \wedge *(R \wedge \hat{\tau})] = \frac{Q}{D^2}\hat{\mathbf{n}} \qquad (17)$$

which is, up to an understandable reinterpretation of notations, exactly the form known as the Coulomb field vector. Here $\hat{\mathbf{n}}^\mu = \hat{\mathbf{D}}^\mu / D \ (\perp \hat{\tau})$ where $R^\mu u'_\mu =: D \equiv \hat{D} := R^\mu \hat{\tau}_\mu$ and $\hat{\mathbf{D}}^\mu = \hat{b}^\mu_\nu R^\nu$ with $\hat{b}^\mu_\nu = \delta^\mu_\nu - \hat{\tau}^\mu \hat{\tau}_\nu$, hence

$$\hat{\mathbf{D}}^\mu = -D^2 a'^\mu - D\,(1 - a'\!\cdot\! R)\, u'^\mu + \frac{1}{2}\left[1 + D^2 a'\!\cdot\! a' + (1 - a'\!\cdot\! R)^2\right] R^\mu, \qquad (18)$$

$\hat{\mathbf{D}}^\mu \neq \mathbf{D}^\mu$; note that $\hat{\mathbf{D}}^\mu \hat{\mathbf{D}}_\mu = -D^2$, as this was the case for $\mathbf{D}^\mu$ in (5). It is clear that $\hat{\mathbf{D}}^\mu + D\hat{\tau}^\mu = R^\mu$.

4.1. Relative three-velocities of reference frames

Let us now simultaneously consider three distinct reference frames and denote them as A, B, and C. Between such frames there can be established quite a few algebraic relations having a clear and important physical meaning, and it is interesting that these relations hold equally in general and special relativity. One defines the relative three-velocity of frame B with respect to frame A (and measured in A) as a (co)vector $\mathbf{v}_{\mathrm{BA}}$ perpendicular to the monad τ_{A}. Let us use the general definition of three velocity as a four vector restricted to the spacelike section corresponding to the reference frame under consideration (A.6) where $\tau_\alpha dx^\alpha / ds = (1 - v^2)^{-1/2}$. Then

$$\tau_{\mathrm{B}} = (\tau_{\mathrm{A}} + \mathbf{v}_{\mathrm{BA}})(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) \text{ and } \mathbf{v}_{\mathrm{BA}}{}^\mu = \frac{\tau^\nu_{\mathrm{B}} b_{\mathrm{A}\nu}^{\ \mu}}{\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}} \qquad (19)$$

(here the relation $\tau^\mu_{\mathrm{B}} - \tau^\mu_{\mathrm{A}}(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) \equiv \tau^\nu_{\mathrm{B}} b_{\mathrm{A}\nu}^{\ \mu}$ was used); hence,

$$\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}} = \frac{1}{\sqrt{1 + \mathbf{v}_{\mathrm{BA}} \cdot \mathbf{v}_{\mathrm{BA}}}} \equiv \frac{1}{\sqrt{1 - \mathbf{v}_{\mathrm{BA}} \bullet \mathbf{v}_{\mathrm{BA}}}} = \frac{1}{\sqrt{1 - \mathbf{v}^2_{\mathrm{BA}}}}. \qquad (20)$$

It is clear that similar relations exist for any pair of reference frames whatever when the respective monads are introduced. We see that there is a symmetry for squared three-velocities between any pair of frames, in particular, $\mathbf{v}^2_{\mathrm{BA}} = \mathbf{v}^2_{\mathrm{AB}}$. Since these three-velocities are described as four-vectors perpendicular to the respective monads (of the frames corresponding to the frame subindex of τ and of b), they belong to different (local) three-spatial sections of spacetime and in general cannot be directly compared by measurements ones with others without further projections onto alternative subspaces. The inevitability of such a situation is quite obvious. Even in the generally used special-relativistic composition-of-velocities formula for globally inertial frames in motion along “same spatial direction,” this is in fact also the case which is tacitly assumed, but frequently not properly understood. Its strict formulation when these velocities are not mutually “parallel,” is however more laborious.

Another useful step in our calculations is to apply same procedure as in (19), but taken with respect to the frames C and A, then to C and B, and further applying it to the free τ_{B}, thus $\tau_{\mathrm{C}} = (\tau_{\mathrm{A}} + \mathbf{v}_{\mathrm{CA}})(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{C}}) = (\tau_{\mathrm{B}} + \mathbf{v}_{\mathrm{CB}})(\tau_{\mathrm{B}} \cdot \tau_{\mathrm{C}}) = [(\tau_{\mathrm{A}} + \mathbf{v}_{\mathrm{BA}})(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) + \mathbf{v}_{\mathrm{CB}}](\tau_{\mathrm{B}} \cdot \tau_{\mathrm{C}})$. When this expression is multiplied by b_{A} under a contraction with the lower (component) index of this factor, we come to

$$\mathbf{v}_{\mathrm{CA}}{}^\nu = \left[\mathbf{v}_{\mathrm{BA}}{}^\nu (\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) + \mathbf{v}_{\mathrm{CB}}{}^\mu b_{\mathrm{A}\mu}^{\ \nu}\right] \frac{\tau_{\mathrm{B}} \cdot \tau_{\mathrm{C}}}{\tau_{\mathrm{A}} \cdot \tau_{\mathrm{C}}}. \qquad (21)$$

In fact, this is the local velocities composition formula $\mathrm{A} \to \mathrm{B} \to \mathrm{C}$ for general (not only inertial) frames in both relativities, special as well as general one. Here, of course, one has to take into account the relation (20). In this paper we do not consider further details of the usual composition formula.

Other relations which are worth being mentioned, are the following ones: those with projections onto the alternative monads,

$$\begin{aligned} \mathbf{v}_{\mathrm{BA}}{}^\nu b_{\mathrm{B}\nu}^{\ \mu} &= -(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) \mathbf{v}_{\mathrm{AB}}{}^\mu, \\ \mathbf{v}_{\mathrm{AB}}{}^\nu b_{\mathrm{A}\nu}^{\ \mu} &= -(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) \mathbf{v}_{\mathrm{BA}}{}^\mu; \end{aligned} \qquad (22)$$

further, due to (19) and (22),

$$\mathbf{v}_{\mathrm{AB}} \cdot \mathbf{v}_{\mathrm{BA}} = -\tau_{\mathrm{A}} \cdot \mathbf{v}_{\mathrm{AB}} = -(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) \mathbf{v}^2_{\mathrm{BA}} = (\tau_{\mathrm{A}} \cdot \mathbf{v}_{\mathrm{AB}})^2 / \mathbf{v}^2_{\mathrm{AB}} \qquad (23)$$

(here the obvious symmetry $\tau_{\mathrm{A}} \cdot \mathbf{v}_{\mathrm{AB}} = \tau_{\mathrm{B}} \cdot \mathbf{v}_{\mathrm{BA}}$ was taken into account); finally,

$$\mathbf{v}_{\mathrm{AB}} = -(\tau_{\mathrm{A}} \cdot \tau_{\mathrm{B}}) \mathbf{v}_{\mathrm{BA}} + (\mathbf{v}_{\mathrm{AB}} \cdot \tau_{\mathrm{A}}) \tau_{\mathrm{A}} \qquad (24)$$

(decomposition with respect to the frame A). Note that $\mathbf{v}^2_{\mathrm{AB}} := \mathbf{v}_{\mathrm{AB}} \bullet \mathbf{v}_{\mathrm{AB}} = -\mathbf{v}_{\mathrm{AB}} \cdot \mathbf{v}_{\mathrm{AB}} > 0$.

Let us globally (at any P) denote in the LW problem the reference frame of inertial observer as A, $\tau_{\mathrm{A}}{}^\mu = \delta^\mu_0$, the retarded frame co-moving with the charge as B, ${\tau_{\rm B}}^{\mu} = u'^{\mu}$, and the frame co-moving with the field, as C (${\tau_{\rm C}}^{\mu} = \hat{\tau}^{\mu}$). Then, on the one hand,

$$(\tau_{\rm B} \cdot \tau_{\rm C}) = (u' \cdot \hat{\tau}) = 1 - \frac{1}{2}\left[D^2 a' \cdot a' + (a' \cdot R)^2\right], \quad (25)$$

and, on the other hand,

$$\mathbf{v}_{\rm CB} = \frac{\hat{\tau}}{(u' \cdot \hat{\tau})} - u'. \quad (26)$$

Rotation of the frame C takes the (not quite easily deducible) form

$$\hat{\omega} = -\frac{1 - D\,(\dot{a}' \cdot R)}{1 - a' \cdot R}\,\mathbf{a}' \times \hat{\mathbf{n}} + D\,\dot{\mathbf{a}}' \times \hat{\mathbf{n}} \quad (27)$$

where 1-form $\dot{a}' = (da'_{\mu}/ds')dx^{\mu}$ describes the retarded third proper-time derivative of position of the charge in its motion along the worldline L. It is worth giving some hints for the deduction of (27): The exterior product of any odd-rank forms α and β is skew-symmetric, thus $\alpha \wedge \alpha \equiv 0$. The vector product $\times$, see (A.5), is applicable to a pair of arbitrary vectors, thus it automatically projects each of them onto the three-dimensional subspace orthogonal to the monad. One now has to apply the general definition of rotation (A.9) to the monad $\tau \Rightarrow \hat{\tau}$. Some simplifications follow immediately. Then to complete the simplification one has to take into account a relation following from the form (not directly from the general definition) of $\hat{\tau}$ (15) and $\hat{\mathbf{D}}$ (18):

$$\hat{\mathbf{D}}_{\mu} = D\hat{\tau}_{\mu} - 2D^2 a'_{\mu} - 2D(1 - a' \cdot R)u'_{\mu} + \left[D^2(a' \cdot a') + (1 - a' \cdot R)^2\right] R_{\mu}$$

(at each subsequent step only very few terms survive). The final result is (27).

Appendix A. Description of reference frames

In this communication we use notations and definitions from [10], see also references therein. A reference frame is understood as the splitting of general four-dimensional physical quantities into parts referred to observer's local time direction and the corresponding local three-dimensional physical subspace orthogonal to it, however the latter (or both parts) are written as four-dimensional tensor quantities (of naturally determined ranks) being orthogonal (or also, if we would wish to emphasize this geometrically, parallel) to observer's time direction. This direction is expressed *via* the unit vector (or covector, the distinction should be understandable from the context, frequently mathematical) τ, the *monad*, tangent to the observer's world line, thus interpreted as the observer's four-velocity at the event (four-dimensional point) where is located the quantity (object) under consideration. Thus we speak about a continuous swarm of observers, a congruence of their world lines without singularities (the lines do not intersect, and through any event goes one and only one such line). The monad and the metric tensor at each event are necessary and sufficient for a complete description of a reference frame. Of course, this presence of a swarm of observers, with all their equipment necessary for measuring of all physical quantities at any event, should not disturb both usual physical fields and (in general relativity) the spacetime geometry (the gravitational field). Here we consider such arbitrary reference frames only in the framework of special relativity, thus the simplest choice of coordinates is Cartesian which we use in this paper. In our treatment reference frames are generally not related to systems of coordinates, and in one and the same system of coordinates any choice of a reference frame (or different choices simultaneously) may be used.

To split spacetime tensors into their above-mentioned parts, two typical projectors are used. A projector is an idempotent, which means that its repeated action automatically reduces to a single action of it, and it differs from the metric tensor possessing a similar (just mentioned) property by the fact that an application of a projector leads to certain partial loss of information. If we describe a projector as a 4×4 matrix (really, a rank two tensor), its determinant should be equal to zero. In more concrete terms, the matrix rank of a projector should be equal to one when we speak about a projector onto a single direction (here, τ), or three when we perform a projection onto the local three-dimensional physical space orthogonal to τ. Thus in the first case we can use the projector

$$\pi^{\mu}_{\nu} = \tau^{\mu}\tau_{\nu} \quad (A.1)$$

and in the second case,

$$b^{\mu}_{\nu} = g^{\mu}_{\nu} - \tau^{\mu}\tau_{\nu}, \quad (A.2)$$

hence

$$\pi^{\mu}_{\lambda}\pi^{\lambda}_{\nu} = \pi^{\mu}_{\nu}, \;\; b^{\mu}_{\lambda}b^{\lambda}_{\nu} = b^{\mu}_{\nu}, \;\; b^{\mu}_{\nu}\pi^{\nu}_{\lambda} = 0, \;\; b^{\mu}_{\nu}\tau^{\nu} = 0. \quad (A.3)$$

However in the first case we frequently use a mere interior multiplication (that is, with a contraction) by τ since this leads to a four-dimensionally well defined quantity. It is also clear that $b^{\mu}_{\nu} + \pi^{\mu}_{\nu} = g^{\mu}_{\nu}$. It is worth being repeated that the matrices corresponding to (A.1) and (A.2) are respectively of ranks one and three.

Traditionally, in the literature one usually finds an implicit identification of a four-dimensional Cartesian system of coordinates and the corresponding ("co-moving") reference frame. This does not pose any ambiguities, only if different reference frames are not considered simultaneously on the background of same system of coordinates, or a non-inertial reference frame is involved. However it is better to take into account that this traditional approach represents a tacit admission that the monad coincides with the unit (timelike) vector along the t-axis and any orthonormal transformation is accompanied with a corresponding change of the monad. There is also a widespread prejudice that non-inertial frames cannot be used in or they contradict to the special theory of relativity, but this is nothing more than a prejudice. In this paper we consider such frames of non-inertial observers in two concrete cases, and the monad approach works perfectly in description of physical situation in these non-inertial frames. We also use another projector (of rank-two matrix, that is, realizing projection onto a two-dimensional subspace) when it simplifies description of the situation, and there should exist a naturally determined spatial direction which enables this description.

It is convenient, in the sense of both calculations and adequate work of physical intuition, to use the vector symbolics of scalar and vector products denoted as $\bullet$ and $\times$. In fact, these operations are coincident with those of the three-dimensional vector algebra, though the objects to which they are applied are four-dimensional vectors restricted to the three-dimensional subspace orthogonal to the monad (not always to the global subspace corresponding in particular to an inertial frame, but, in rotating frames, changing to the more general local non-holonomic case: see in the end of this appendix comments related to the three-dimensional subspaces then having such a local meaning only). These products are defined as

$$\mathbf{p} \bullet \mathbf{q} := -b_{\mu\nu} p^\mu q^\nu \equiv *[(\tau \wedge p) \wedge *(\tau \wedge q)] \tag{A.4}$$

and

$$\mathbf{p} \times \mathbf{q} = *(p \wedge \tau \wedge q). \tag{A.5}$$

We use here the Cartan exterior forms notations such as the wedge product $\wedge$, the Hodge star operation $*$ (the dual conjugation of a p-form, not necessarily of a 2-form = skew-symmetric rank-two tensor), and, later, the exterior differential d, see for details and references [10].

In Cartesian coordinates, due to the spacetime signature $(+,-,-,-)$, the monad of the frame co-moving with these coordinates is $\tau^\mu = \delta^\mu_0$, $\tau_\mu = \delta^0_\mu$. Thus (A.5) becomes $(\mathbf{p} \times \mathbf{q})^i = \epsilon_{ijk} p^j q^k$. The (co)vectors lying in the three-dimensional subspace of a reference frame are usually written as four-dimensional ones, but in some important cases we put them in boldface printing (as $\mathbf{E}$ and $\mathbf{B}$ for electric and magnetic vectors). Then $\mathbf{E}^2 \equiv \mathbf{E} \bullet \mathbf{E} = -b_{\mu\nu} \mathbf{E}^\mu \mathbf{E}^\nu$, *etc.*

The three-dimensional velocity $\mathbf{v}$ (described as a four-vector $\perp$ τ) of a pointlike particle from the viewpoint of reference frame corresponding to the monad τ, is determined *via* the splitting of its four-velocity $u^\mu = dx^\mu/ds$,

$$u = (\tau \cdot u)(\tau + \mathbf{v}), \qquad \text{or equivalently } \mathbf{v}^\mu = b^\mu_\nu \frac{dx^\nu}{\tau_\alpha dx^\alpha} \tag{A.6}$$

where $\tau_\alpha dx^\alpha/ds = (1-v^2)^{-1/2}$; *cf.* also (20) and the corresponding remarks. This is, of course, an exclusion in the general method of projecting vector and tensor quantities. Another exclusion is the relation between the four-dimensional acceleration and its usual three-dimensional counterpart which is applied in making an easier comparison with the Landau–Lifshitz treatment of the LW field [1]. It is now convenient to write the corresponding relations in the (local) three-dimensional subspace notations. The relativistic acceleration four-vector then is

$$a'^\mu = \frac{du'^\mu}{ds'} = \frac{1}{\sqrt{1-\mathbf{v}'^2}} \left[\frac{d}{dt'}\left(\frac{1}{\sqrt{1-\mathbf{v}'^2}}\right)\right](1, \mathbf{v}') + \frac{1}{1-\mathbf{v}^2}\left(0, \dot{\mathbf{v}}'\right),$$

and the orthogonality of a' and u',

$$u' \cdot a' = \frac{d}{dt'}\left(\frac{1}{\sqrt{1-\mathbf{v}'^2}}\right) - \frac{1}{(1-\mathbf{v}'^2)^{3/2}} \mathbf{v}' \bullet \dot{\mathbf{v}}' = 0, \tag{A.7}$$

finally yields a simpler relation between the four- and three-acceleration

$$a'^\mu = \frac{\mathbf{v}' \bullet \dot{\mathbf{v}}'}{(1-\mathbf{v}'^2)^2}(1, \mathbf{v}') + \frac{1}{1-\mathbf{v}'^2}\left(0, \dot{\mathbf{v}}'\right). \tag{A.8}$$

Rotation of a reference frame is defined as

$$\begin{aligned} \omega &= *(\tau \wedge d\tau) \equiv 2 * (\tau \wedge A), \\ A &= \frac{1}{2} A_{\mu\nu} dx^\mu \wedge dx^\nu, \end{aligned} \tag{A.9}$$

while in Cartesian coordinates and with τ describing a non-inertial frame, A (not the electromagnetic four-potential 1-form, but the rotation 2-form) is the skew term in the natural decomposition of gradient of the monad,

$$\begin{aligned} \tau_{\mu,\nu} &= \tau_\nu G_\mu + A_{\nu\mu} + D_{\nu\mu}, \\ A_{\mu\nu} &= A_{[\mu\nu]}, \quad D_{\mu\nu} = D_{(\mu\nu)}, \end{aligned} \tag{A.10}$$

G being acceleration of the reference frame and D, the frame's symmetric rate-of-strain tensor; G, A, and D belong to the above-mentioned three-dimensional (local) subspace. Of course, all these quantities become equal to zero in any inertial frame globally. When $A \neq 0$ (equivalent to $\omega \neq 0$), the three-dimensional subspace orthogonal to τ is non-holonom, that is, there only exists an overall distribution of elements of the corresponding (now non-holonom) hypersurface, but these elements do not fit together to form a global spatial hypersurface in the proper (holonom) sense, see [10], the fact well known in geometry of congruences (here we are dealing with the τ-congruence).

Appendix B. Electromagnetic fields in arbitrary reference frames

Let us now apply the definitions given in Appendix A. to the electromagnetic field and related quantities. The field tensor $F_{\alpha\beta}$ which also can be written as a 2-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{B.1}$$

splits into two four-dimensional vectors, electric

$$\mathbf{E}_\mu = F_{\mu\nu}\tau^\nu \iff \mathbf{E} = *(\tau \wedge *F) \tag{B.2}$$

and magnetic

$$\mathbf{B}_\mu = -\overset{*}{F}_{\mu\nu}\tau^\nu \iff \mathbf{B} = *(\tau \wedge F), \tag{B.3}$$

both $\perp \tau$, see also (1); 2-form $F := \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. This splitting follows from an observation that the Lorentz force can be expressed as

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B})_\alpha = F_{\mu\nu}\left(\tau^\nu + \mathbf{v}^\nu\right) b^\mu_\alpha. \tag{B.4}$$

In Cartesian coordinates (and with the corresponding inertial monad) we have the same relations as for usual contravariant three-vectors:

$$\begin{aligned} \mathbf{E}^i &= F_{i0} = -F^{i0}, \\ \mathbf{B}^i &= -\frac{1}{2}\epsilon_{ijk}F_{jk} = -\frac{1}{2}\epsilon_{ijk}F^{jk}, \end{aligned} \tag{B.5}$$

thus

$$F_{ij} = F^{ij} = -\epsilon_{ijk}\mathbf{B}^k. \tag{B.6}$$

The electromagnetic stress-energy tensor is

$$T_{\mathrm{em}}{}_\mu^\nu = \frac{1}{4\pi}\left(\frac{1}{4}F_{\kappa\lambda}F^{\kappa\lambda}\delta^\nu_\mu - F_{\mu\lambda}F^{\nu\lambda}\right) \tag{B.7}$$

(in Gaussian units). Its deduction is most simple when one considers Maxwell's equations in tensor form in a vacuum and without sources. Its (single) contraction with arbitrary monad includes the Poynting vector in that frame,

$$T_{\mathrm{em}}{}_\mu^\nu \tau_\nu = \frac{1}{8\pi}\left[\left(\mathbf{E}^2 + \mathbf{B}^2\right)\tau_\mu + 2(\mathbf{E}\times\mathbf{B})_\mu\right], \tag{B.8}$$

and the squared expression is (*cf.* [12, 13, 16])

$$\begin{aligned} T_{\mathrm{em}}{}_\mu^\nu T_{\mathrm{em}}{}_\xi^\mu \tau_\nu \tau^\xi &= \frac{1}{(8\pi)^2}\left[\left(\mathbf{E}^2 + \mathbf{B}^2\right)^2 - 4(\mathbf{E}\times\mathbf{B})^2\right] \\ &\equiv \frac{1}{(8\pi)^2}\left[\left(\mathbf{B}^2 - \mathbf{E}^2\right)^2 + 4(\mathbf{E}\bullet\mathbf{B})^2\right] \\ &= \frac{1}{(16\pi)^2}\left({I_1}^2 + {I_2}^2\right) \end{aligned} \tag{B.9}$$

(it is interesting that this expression is not only a scalar under transformations of coordinates, but it is also independent of the choice of reference frame: the right-hand side does not involve any mention of the monad at all). For the LW field [due to (9)] this takes a very concise form,

$$T_{\mathrm{em}}{}_\mu^\nu T_{\mathrm{em}}{}_\xi^\mu \tau_\nu \tau^\xi = \left(\frac{Q^2}{8\pi D^4}\right)^2. \tag{B.10}$$

Finally, it is worth mentioning that the three velocity of propagation of any electromagnetic field with respect to a given reference frame τ is

$$\mathbf{v} = 2\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{E}^2 + \mathbf{B}^2}. \tag{B.11}$$

Another comment on physical interpretation of the Poynting vector is that it does not always describe propagation of extractable energy of the field (see also [15]); the exclusion is here related to the special case of static and stationary fields (whose frequency is equal to zero) which, by the way, also manifest an exotic kind of the Doppler effect reduced to an intensity change only.

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