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Planar Electrons from First Principles

Contents

Abstract

In this work we construct spinors describing planar electrons (”spin” 1/2 particles with well defined parity) in $d = 1 + 2$ from first principles and show that they satisfy Dirac equation, which turns out to be the covariant form of the eigenvalue equation for intrinsic parity just like in $d = 1 + 3$. We stick to the conventional definition of parity as space inversion. In this picture, the fermion mass term is invariant under parity and time reversal. An explicit construction of charge conjugation operator, and a non-relativistic expansion of the gauged Dirac equation shows that ”spin” 1/2 planar fermions have fixed ”spin” projection.

1. Introduction

The quantum description of physical systems necessarily entails a Hilbert space with a positive defined inner product. Invariance of probability amplitudes requires the symmetry operations on the system to be represented by unitary or anti-unitary operators acting on the Hilbert space [1]. Thus, the states themselves must carry a unitary representation of the symmetry group. If in addition we are dealing with an elementary system, the states must belong to the unitary irreducible representations of the symmetry group. This is what we will mean in the following by first principles.

Over the past few years the interest on theories in extra dimensions exponentially growth due to the possibility of the existence of large extra dimensions left open by the experimentally poorly tested quantum effects of gravity [2]. The large amount of existing work on the subject, heavily rests on the properties of the gamma matrices $\gamma^\mu, \mu = 0, 1, ..., d-1$ spanning the Clifford algebra (CA) in higher dimensions [3, 4]. In this construction, the odd-dimensional case is distinguished from the even-dimensional one by the fact that $\gamma \equiv \gamma^0\gamma^1...\gamma^{d-1}$is an element of the CA in the former case. Indeed, it can be shown [4] that γ anti-commute with γ^μ in the even d case, but commute for odd d . This can be used to show that γ is proportional to γ^{d-1} with the proportionality constant depending on the dimension of space -time as $\gamma^{d-1} = \pm\gamma$ for $d = 1$ mod 4 and $\gamma^{d-1} = \pm i\gamma$ for $d = 3$ mod 4. Thus, Clifford algebra in odd dimensions has two inequivalent representations depending on the chosen sign. In contrast, in even dimensions there is only one representation.

The transformation properties of Dirac fields under $\mathbb{C}$, $\mathbb{P}$ and $\mathbb{T}$ in this formalism where discussed in [3, 4]. In particular, the definition of parity changes in odd-dimensions where it is defined as the reversing of all but one of the spatial components. This is due to the fact that the generalization of the conventional parity, i.e. the reversing of all space coordinate turns out to be a *proper* (det=1) transformation in odd d, in contrast to the even d case where parity is an improper transformation. The study of the transformation properties of bi-linears under discrete symmetries yields a fermion mass term which violates the so-defined parity and charge conjugation for $d = 5$ mod 4 and parity and time reversal for $d = 3$ mod 4.

In this work we study in detail "spin" 1/2 particles in $d = 1 + 2$ from first principles. We recalculate the irreps of the proper orthochronous Lorentz subgroup ($L_+^\uparrow$) of HLG and explicitly construct spinors for the lowest dimensional irrep. We implement discrete transformations according to the group properties. We show that if we follow the current trend of considering parity as an improper transformation, then spinors describing "spin" 1/2 particles in $d = 1 + 2$ are not parity eigenstates. In contrast, if we deviate from the usual convention and define parity in the same way as in $d = 1 + 3$, i.e., as the reversing of all the spatial coordinates (spatial inversion in the following), then Dirac equation is just the covariant form of the eigenvalue equation for intrinsic spatial inversion just like in $d = 1 + 3$, in spite of the fact that under this definition spatial inversion is a proper transformation. In this picture, the fermion mass term is invariant under spatial inversion and time reversal in sharp contrast with results obtained under the definition of parity in the current literature [3, 4]. Finally we construct the charge conjugation operator and show that the eigenstates of spatial inversion (Dirac fields) are connected by charge conjugation just like in $d = 1 + 3$. Thus the particle has fixed "spin" up while antiparticle carries a fixed "spin" down. This conclusion is reinforced by a non-relativistic expansion of the gauged equation which explicitly shows that, in the non-relativistic limit, only the up component of Dirac field survives, i.e. a non-relativistic particle has a singled valued spin projection.

2. Poincaré group and discrete symmetries in $d = 1 + 3$

In this section we briefly review the structure of the Poincaré group in 1+3 dimensions emphasizing the formal structure of discrete symmetries.

2.1. Poincaré transformations in $d = 1 + 3$: brief review

The most general transformation leaving the space-time interval invariant is

$$x' = \Lambda x + a = \mathcal{L}(a, \Lambda)x, \tag{1}$$

where a is a constant four-vector and Λ a real 4×4 matrix satisfying

$$\Lambda^T \eta \Lambda = \eta. \tag{2}$$

These transformations constitute a group, the Poincaré group, which we will denote $\mathbf{P}(1, 3)$, with the composition law

$$\mathcal{L}(\overline{a}, \bar{\Lambda})\mathcal{L}(a, \Lambda) = \mathcal{L}(\bar{\Lambda}a + \overline{a}, \bar{\Lambda}\Lambda). \tag{3}$$

The Poincaré group is a semi-direct product of the translation group, $\mathbf{T}^{(1,3)}$, and the Homogeneous Lorentz Group (HLG)

$$\mathbf{P}(1, 3) = \mathbf{T}^{(1,3)} \otimes HLG, \tag{4}$$

spanned by the transformations Λ. The global structure of the HLG is composed of four disjoint sets characterized by $\det\Lambda = \pm 1$ and $\Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$ according to

1. *Proper orthochronous*, set denoted $L_+^\uparrow$ ($\det\Lambda = 1,\ \Lambda^0{}_0 \geq +1$);
2. *Improper orthochronous*, $L_-^\uparrow$ ($\det\Lambda = -1,\ \Lambda^0{}_0 \geq +1$);
3. *Proper non-orthochronous*, $L_+^\downarrow$ ($\det\Lambda = 1,\ \Lambda^0{}_0 \leq -1$);
4. *Improper non-orthochronous*, $L_-^\downarrow$ ($\det\Lambda = -1,\ \Lambda^0{}_0 \leq -1$).

In this decomposition, only $L_+^\uparrow$ constitute a subgroup.

2.1.1. Cosets and discrete symmetries

The sets $L_-^\uparrow, L_+^\downarrow, L_-^\downarrow$ are not subgroups of the HLG but rather cosets built in terms of the discrete elements of HLG. Indeed, in Minkowski space there are only two such independent elements: space inversion (parity transformation) and time inversion (time reversal). Space inversion is the *improper* transformation changing the sign of space coordinates and has the representation

$$\mathcal{P} = \mathrm{diag}(1, -1, -1, -1), \tag{5}$$

This operator satisfies

$$\det \mathcal{P} = -1, \quad \mathcal{P}^0{}_0 = 1. \quad \mathcal{P}^2 = \mathbf{1}_4, \tag{6}$$

thus parity belongs to $L_-^\uparrow$. Furthermore, every transformation $\Lambda_-^\uparrow$ in $L_-^\uparrow$ can be written as $\Lambda_-^\uparrow = \mathcal{P}\Lambda_+^\uparrow$ for some $\Lambda_+^\uparrow$ in $L_+^\uparrow$, hence $L_-^\uparrow$ is a coset constructed from parity: $L_-^\uparrow = \mathcal{P}L_+^\uparrow$.

In a similar way time reversal is the *improper* transformation changing the sign of the time coordinate and has the following representation in Minkowski space

$$\mathcal{T} = \mathrm{diag}(-1, 1, 1, 1). \tag{7}$$

It satisfies

$$\det \mathcal{T} = -1, \quad \mathcal{T}^0{}_0 = -1, \quad \mathcal{T}^2 = \mathbf{1}_4. \tag{8}$$

This transformation belongs to $L_-^\downarrow$ and the whole latter set is actually a coset constructed from time reversal: $L_-^\downarrow = \mathcal{T}L_+^\uparrow$.

Finally $L_+^\downarrow$ is the coset constructed from total reflection ($\mathcal{R}_T$), a *proper* discrete symmetry which can be written in terms of $\mathcal{P}$ and $\mathcal{T}$ as $\mathcal{R}_T = \mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P} = -\mathbf{1}_4$. Thus $L_+^\downarrow = \mathcal{R}_T L_+^\uparrow$.

We remark that *in 1+3 dimensional Minkowski space the discrete transformations $\mathcal{P}, \mathcal{T}$ and $\mathcal{R}_T$ are the operations connecting the four disjoint sets into which the HLG can be decomposed.*

2.2. Quantum Poincaré transformations in $d = 1 + 3$: brief review

In the previous section we analyzed Poincaré transformations at the *classical* level, i.e. on Minkowski space. In this section we turn to the *quantum* case. According to the quantum principles the characterization of Hilbert space must be done in terms of the symmetries of the corresponding system. In other words, a system lives in one of the irreducible representations (irreps) of its symmetry group. For a free particle this amounts to the decomposition of Hilbert space into irreps of the Poincaré group. In the conventional approach, Poincaré irreps are labeled by the eigenvalues m^2 and $-s(s+1)p^2$ of its two casimir operators, P^2 and the squared Pauli-Lubanski operator W^2 respectively. Thus a free elementary system is characterized by its mass and spin and belongs to the $2s+1$-dimensional spin space which will denote $\{m, s\}$. However, the formulation of manifestly covariant field theories require the basic fields to transform as a linear representation of the Homogenous Lorentz group. The field equations and subsidiary conditions (if necessary) must then be chosen so as to recover the one-particle Poincaré group representation. The discrete symmetries are imposed a posteriori on the fields and the corresponding operators explicitly constructed. A more modern viewpoint [5] starts from the implementation of discrete symmetries in Hilbert space and later exploit the consequences for the irreps of the Poincaré group. Here we follow this path and later we generalize it to $d \neq 4$ emphasizing the role of global properties of the group.

Firstly we briefly review the implementation of Poincaré transformations in Hilbert space. We follow closely Ref. [5] except for the metric of space-time. We start with transformations $\mathcal{L}(a, \Lambda) \in \mathbf{P}(1, 3)$, where $\Lambda \in L_+^\uparrow$. This is the part of the Pioncaré group connected to the identity. These transformations induce linear unitary operations in Hilbert space, implemented by the operators $U(a, \Lambda)$

$$|\psi\rangle \longrightarrow U(a, \Lambda)|\psi\rangle, \tag{9}$$

satisfying the composition law

$$U(\overline{a}, \overline{\Lambda})U(a, \Lambda) = U(\overline{\Lambda}a + \overline{a}, \overline{\Lambda}\Lambda) \tag{10}$$

and the factorization

$$U(a, \Lambda) = U(a, 1)U(0, \Lambda). \tag{11}$$

These operators can be written in exponential form

$$U(a, \Lambda) = \exp\left(-\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} + ia_\mu P^\mu\right), \tag{12}$$

and in particular

$$\begin{aligned} U(a, 1) &= \exp(ia_\mu P^\mu), \\ U(0, \Lambda) &= \exp\left(-\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}\right), \end{aligned} \tag{13}$$

where $J^{\mu\nu}$ is the anti-symmetric tensor containing the angular momentum $\mathbf{J}$ and boost $\mathbf{K}$ operators, and P^μ is the 4-momentum operator.

Considering infinitesimal transformations in Hilbert and Minkowski space it is easy to show that the following relations hold

$$\begin{aligned} &[P^\mu, P^\rho] = 0, \\ &[J^{\rho\sigma}, P^\mu] = i\left(\eta^{\sigma\mu}P^\rho - \eta^{\rho\mu}P^\sigma\right), \\ &[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} \\ &\qquad\qquad -\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho}). \end{aligned} \tag{14}$$

Taking $H = P^0$ we see that $\mathbf{P} \equiv (P^1, P^2, P^3)$ and $\mathbf{J} \equiv (J^{23}, J^{31}, J^{12})$ are conserved quatities. In contrast the boost generator $\mathbf{K} \equiv (J^{01}, J^{02}, J^{03})$ is not conserved.

In terms of these operators the commutation relations read

$$
\begin{aligned}
[H,H] &= [P_i,H] = [J_i,H] = 0,\\
[K_i,H] &= -iP_i,\\
[J_i,P_j] &= i\epsilon_{ijk}P_k,\\
[K_i,P_j] &= -iH\delta_{ij},\\
[J_i,J_j] &= i\epsilon_{ijk}J_k,\\
[J_i,K_j] &= i\epsilon_{ijk}K_k,\\
[K_i,K_j] &= -i\epsilon_{ijk}J_k.
\end{aligned}
\tag{15}
$$

Strictly speaking these are the commutation relations of the part of the Poincaré group connected to the identity, namely $\mathbf{T}^{(1,3)} \otimes L_+^{\uparrow}$.

Following the semi-direct structure of the Poincaré group the characterization of Hilbert space is done in terms of irreps of the proper orthochronous Lorentz group, the piece of HLG connected to the unity. This can be easily done due to the isomorphism between the proper orthochronous Lorentz Group and the $SU(2)_A \otimes SU(2)_B$ group spanned by the generators

$$
\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \qquad \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}), \tag{16}
$$

which allows us to label the irreps of this group by two $SU(2)$ labels (a,b) associated with the Casimir operators $\mathbf{A}^2, \mathbf{B}^2$, of $SU(2)_A$ and $SU(2)_B$ respectively. States in these irreps are $|a,m_a;b,m_b\rangle \equiv |a,m_a\rangle \otimes |b,m_b\rangle$ where $|a,m_a\rangle$, $|b,m_b\rangle$ are the $SU(2)_A$ and $SU(2)_B$ eigenstates respectively

$$
\begin{aligned}
\mathbf{A}^2|a,m_a\rangle &= a(a+1)|a,m_a\rangle,\\
A_3|a,m_a\rangle &= m_a|a,m_a\rangle,\\
\mathbf{B}^2|b,m_b\rangle &= b(b+1)|b,m_b\rangle,\\
B_3|b,m_b\rangle &= m_b|a,b\rangle.
\end{aligned}
$$

This structure allow us to explicitly construct operators for rotations and boosts, in an arbitrary representation (a,b), in terms of the representations of the operators $\mathbf{A}$, $\mathbf{B}$. We remark that these irreps in general are not irreps of the Poincaré group. Indeed, whereas the irreps of the Poincaré group (more precisely the piece of the PG connected to the identity) are $2s+1$-dimensional subspaces, the irreps of the proper orthochronous Lorentz group are $(2a+1)\times(2b+1)$-dimensional subspaces which we will denote as $\{m,(a,b)\}$. The Poincaré content in the irreps of the proper orthcochronous Lorentz group is

$$
\begin{aligned}
\{m,(a,b)\} \rightarrow \{m, s = |a-b|\} \oplus \\
\{m, s=|a-b|+1\} \oplus ... \oplus \{m, s = a+b)\}.
\end{aligned}
$$

The only case when the irreps of PG and the proper orthochronous Lorentz group coincide is when one of the quantum numbers a or b vanishes. The representation $(a,0)$ is such that $\mathbf{A} = \mathbf{J} = i\ \mathbf{K}$ and we can set $a = s$ since in this case the $SU(2)_A$ coincides with the rotation subgroup. Similarly, for the representation $(0,b)$ we have $\mathbf{B} = \mathbf{J} = -i\ \mathbf{K}$ and we can also set $b = s$ since $SU(2)_B$ coincides with the rotation subgroup in this case. For historical reasons these representations are called "right" and "left" representations respectively. The boost and rotation operators are particularly simple for these representations due to the relations $\mathbf{A} = \mathbf{J} = i\ \mathbf{K}$ and $\mathbf{B} = \mathbf{J} = -i\ \mathbf{K}$. The corresponding boost operators read

$$
B_R(\varphi) = \exp(\mathbf{J}\cdot\mathbf{n}\varphi), \quad B_L(\varphi) = \exp(-\mathbf{J}\cdot\mathbf{n}\varphi).
$$

Now, turning to the general case, quantum states are completely characterized and a representation can be explicitly given in terms of the basis $|a,m_a;b,m_b\rangle$. For a given representation (a,b) we have the representations for $\mathbf{A}$, $\mathbf{B}$ and we construct the generators $\mathbf{J}$, $\mathbf{K}$ from Eqs. (16). A general transformation in $L_+^{\uparrow}$ can be written as

$$
U_\Lambda(\theta,\varphi) = \exp(-i(\mathbf{J}\cdot\theta + \mathbf{K}\cdot\varphi)).
$$

In particular taking $\theta = 0$ we get the operator for boosts which can be used to give a representation of an arbitrary state just acting upon the corresponding state in the rest frame. This way we construct states in Hilbert space from first principles without referring to a given equation of motion. Of course, concerning physics, all we know from these states is that they have spin according to Eq.(**??**). However, in general we could be interested in states with some other well defined properties such as parity. Next we study discrete symmetries and later we will explicitly construct states with well defined spin and parity according to the procedure just outlined.

2.2.1. Quantum discrete symmetries for $d = 1+3$

Since discrete symmetries belong to HLG we require the composition law in Eq.(10) also hold for them. Thus there must exist operators [5]

$$
\Pi \equiv U(0,\mathcal{P}), \quad \Theta \equiv U(0,\mathcal{T}) \quad \mathbb{R}_T \equiv U(0,\mathcal{R}_\mathcal{T}) \tag{17}
$$

satisfying

$$
\begin{aligned}
\Pi U(a,\Lambda)\Pi^{-1} &= U(\mathcal{P}a, \mathcal{P}\Lambda\mathcal{P}^{-1}),\\
\Theta U(a,\Lambda)\Theta^{-1} &= U(\mathcal{T}a, \mathcal{T}\Lambda\mathcal{T}^{-1}),\\
\mathbb{R}_T U(a,\Lambda)\mathbb{R}_T^{-1} &= U(\mathcal{R}_\mathcal{T}a, \mathcal{R}_\mathcal{T}\Lambda\mathcal{R}_\mathcal{T}^{-1})
\end{aligned}
\tag{18}
$$

for every a and $\Lambda \in L_+^{\uparrow}$.

Considering infinitesimal transformations in Hilbert space we obtain the transformation properties of the Poincaré group generators under discrete transformations as

$$\begin{aligned}
\Pi\,(iJ^{\mu\nu})\,\Pi^{-1} &= i\mathcal{P}_{\rho}{}^{\mu}\mathcal{P}_{\sigma}{}^{\nu}J^{\rho\sigma},\\
\Pi\,(iP^{\mu})\,\Pi^{-1} &= i\mathcal{P}_{\rho}{}^{\mu}P^{\rho},\\
\Theta\,(iJ^{\mu\nu})\,\Theta^{-1} &= i\mathcal{T}_{\rho}{}^{\mu}\mathcal{T}_{\sigma}{}^{\nu}J^{\rho\sigma},\\
\Theta\,(iP^{\mu})\,\Theta^{-1} &= i\mathcal{T}_{\rho}{}^{\mu}P^{\rho},\\
\mathbb{R}_T\,(iJ^{\mu\nu})\,\mathbb{R}_T^{-1} &= i\mathcal{R}_{T\,\rho}{}^{\mu}\mathcal{R}_{T\,\sigma}{}^{\nu}J^{\rho\sigma},\\
\mathbb{R}_T\,(iP^{\mu})\,\mathbb{R}_T^{-1} &= i\mathcal{R}_{T\,\rho}{}^{\mu}P^{\rho}.
\end{aligned} \tag{19}$$

We did not cancelled the i factors in these equations since we must first answer the question whether the corresponding transformations are linear and unitary or not. Taking $\mu = 0$ in Eqs. (19) we obtain

$$\begin{aligned}
\Pi\,(iH)\,\Pi^{-1} &= iH,\\
\Theta\,(iH)\,\Theta^{-1} &= -iH,\\
\mathbb{R}_T\,(iH)\,\mathbb{R}_T{}^{-1} &= -iH
\end{aligned} \tag{20}$$

It can be easily shown that in order to avoid the existence of negative energy states we must choose *space inversion operator* Π as **unitary** and *time reversal* Θ *and total inversion* $\mathcal{R}_T$ *operators* as **anti-unitary**. Under this realization

$$\begin{aligned}
\Pi H\Pi^{-1} &= H,\\
\Theta H\Theta^{-1} &= H\\
\mathbb{R}_T H\mathbb{R}_T{}^{-1} &= H.
\end{aligned} \tag{21}$$

Using these results, from Eqs. (19) we get the following transformation properties for $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ under discrete transformations

$$\begin{aligned}
&\Pi\,\mathbf{P}\,\Pi^{-1} = -\mathbf{P}, && \Pi\,\mathbf{J}\,\Pi^{-1} = +\mathbf{J},\\
&\Pi\,\mathbf{K}\,\Pi^{-1} = -\mathbf{K}, && \Theta\,\mathbf{P}\,\Theta^{-1} = -\mathbf{P},\\
&\Theta\,\mathbf{J}\,\Theta^{-1} = -\mathbf{J}, && \Theta\,\mathbf{K}\,\Theta^{-1} = +\mathbf{K}.\\
&\mathbb{R}_T\,\mathbf{P}\,\mathbb{R}_T^{-1} = +\mathbf{P}, && \mathbb{R}_T\,\mathbf{J}\,\mathbb{R}_T^{-1} = -\mathbf{J},\\
&\mathbb{R}_T\,\mathbf{K}\,\mathbb{R}_T^{-1} = -\mathbf{K}.
\end{aligned} \tag{22}$$

3. Dirac equation and spatial inversion in $d = 1 + 3$

Let us now consider spatial inversion for an arbitrary Poincaré irrep. We can track the action of spatial inversion in an arbitrary Poincaré irrep from its action on the irreps of the proper orthochronous Lorentz group. Indeed, from Eqs. (22,16) we see that the generators of $SU(2)_A \otimes SU(2)_B$ transform under spatial inversion as $\mathbf{A} \longrightarrow \Pi\mathbf{A}\Pi^{-1} = \mathbf{B}$, $\mathbf{B} \longrightarrow \Pi\mathbf{B}\Pi^{-1} = \mathbf{A}$, i.e. under this transformation, $(a, b) \leftrightarrow (b, a)$. Thus, the irreps of $L_+^{\uparrow}$, (a, b), are not irreps of the full ortochronous group $L^{\uparrow}$, except for $a = b$. The subspaces of Hilbert space invariant under boosts, rotations and spatial inversion, i.e. the irreps of $L^{\uparrow}$ are actually $(a, b) \oplus (b, a)$.

An important property of these irreps of $L^{\uparrow}$ concerns the role of spatial inversion in the construction of the scalar product. This is essentially due to our choice of finite dimensional representation which requires non-unitary boost operators. Indeed, from Eqs. (16) we get $\mathbf{K}^{\dagger} = -\mathbf{K}$, which yields a self-conjugate boost operator. A general $L_+^{\uparrow}$ transformation

$$U_{\Lambda}(\theta, \varphi) = \exp(-i(\mathbf{J}\cdot\theta + \mathbf{K}\cdot\varphi)),$$

satisfy

$$\Pi[U_{\Lambda}(\theta, \varphi)]^{\dagger}\Pi = [U_{\Lambda}(\theta, \varphi)]^{-1}. \tag{23}$$

In particular for boosts ($\theta = 0$) we obtain

$$\Pi[B(\varphi)]^{\dagger}\Pi = [B(\varphi)]^{-1}, \tag{24}$$

where $B(\varphi) \equiv U_{\Lambda}(\mathbf{0}, \varphi)$. Using Eq.(23) it is easy to show that the product

$$(\psi, \phi) \equiv \langle\psi|\Pi|\phi\rangle.$$

is indeed invariant under transformations of the orthochronous Lorentz group $L^{\uparrow}$.

Let us now consider the specific case of the spinorial representation, $(1/2, 0) \oplus (0, 1/2)$, usually called Dirac representation. In the basis $\{|1/2, \pm1/2\rangle_R \oplus |1/2, \pm1/2\rangle_L\}$, the boost operator for this irrep of $L^{\uparrow}$ reads

$$B(\varphi) = \begin{pmatrix} B_R(\varphi) & 0 \\ 0 & B_L(\varphi) \end{pmatrix} = \begin{pmatrix} \cosh(\varphi/2) + \sigma\cdot\mathbf{n}\sinh(\varphi/2) & 0 \\ 0 & \cosh(\varphi/2) - \sigma\cdot\mathbf{n}\sinh(\varphi/2) \end{pmatrix}.$$

Now, for rest frame states, the rapidity φ can be written in terms of the energy and momentum of the particle as seen in the boosted frame as

$$\begin{aligned}
\cosh\varphi &= \frac{E}{m}, \qquad \sinh\varphi = \frac{|\mathbf{p}|}{m},\\
\cosh\frac{\varphi}{2} &= \sqrt{\frac{E+m}{2m}},\\
\sinh\frac{\varphi}{2} &= \frac{|\mathbf{p}|}{\sqrt{2m(E+m)}}
\end{aligned} \tag{25}$$

and the boost operator reads

$$B(\mathbf{p}) = N\begin{pmatrix} E + m + \sigma\cdot\mathbf{p} & 0 \\ 0 & E + m - \sigma\cdot\mathbf{p} \end{pmatrix}, \tag{26}$$

where $N = 1/\sqrt{2m(E+m)}$. On the other hand, from the transformation properties of $\mathbf{A}$, $\mathbf{B}$ under spatial inversion we find the most general representation of this operator as

$$\Pi = \begin{pmatrix} 0 & a \\ a^{*} & 0 \end{pmatrix}, \tag{27}$$

where a is a phase.

Let us now consider the eigenvalue equation for spatial inversion in the rest frame and in momentum space

$$\Pi\psi(\mathbf{0}) = \pi\psi(\mathbf{0}), \tag{28}$$

where π is the corresponding eigenvalue which is restricted to $\pi = \pm 1$ if we require $\mathbf{\Pi}^2$ to leave strictly invariant the $(1/2, 0) \oplus (0, 1/2)$ space. Solving this eigenvalue equation and setting the phase to $a = 1$ we obtain a specific representation for the states in the rest frame as

$$\psi_+^+(\mathbf{p}) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \psi_-^+(\mathbf{p}) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$
$$\psi_+^-(\mathbf{p}) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \psi_-^-(\mathbf{p}) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{29}$$

where the superindex denotes the spatial inversion eigenvalue $\pi = \pm 1$ and the subindex denotes the J_3 eigenvalue $m = \pm 1/2$. Acting upon these states with the boost operator we obtain the states describing spin 1/2 particles of momentum $\mathbf{p}$ with well defined properties under spatial inversion (electrons for $\pi = 1$, positrons for $\pi = -1$) and spin projection, from first principles and without invoking any equation of motion

$$\psi_+^+(\mathbf{p}) = \frac{N}{\sqrt{2}} \begin{pmatrix} E+m+p_z \\ p_+ \\ E+m-p_z \\ -p_+ \end{pmatrix},$$
$$\psi_+^-(\mathbf{p}) = \frac{N}{\sqrt{2}} \begin{pmatrix} p_- \\ E+m-p_z \\ -p_- \\ E+m+p_z \end{pmatrix},$$
$$\psi_-^+(\mathbf{p}) = \frac{N}{\sqrt{2}} \begin{pmatrix} E+m+p_z \\ p_+ \\ -(E+m-p_z) \\ p_+ \end{pmatrix},$$
$$\psi_-^-(\mathbf{p}) = \frac{N}{\sqrt{2}} \begin{pmatrix} p_- \\ E+m-p_z \\ p_- \\ -(E+m+p_z) \end{pmatrix}. \tag{30}$$

where $p_\pm = p_x \pm i p_y$. Now, these states are traditionally constructed as solutions of the Dirac equation. The link to this equation can be easily established by boosting the parity eigenvalue equation (28) and using Eq. (24). We obtain

$$\left[B^2(\mathbf{p})\Pi - \pi\right]\psi(\mathbf{p}) = 0. \tag{31}$$

The squared boost operator reads

$$B^2(\mathbf{p}) = \begin{pmatrix} \frac{E+\sigma\cdot\mathbf{p}}{m} & 0 \\ 0 & \frac{E-\sigma\cdot\mathbf{p}}{m} \end{pmatrix}$$

Using this operator in Eq.(31) we get

$$\left[B^2(\mathbf{p})\Pi - \pi\right]\psi(\mathbf{p}) = \begin{pmatrix} \pi & \frac{E-\sigma\cdot\mathbf{p}}{m} \\ \frac{E+\sigma\cdot\mathbf{p}}{m} & \pi \end{pmatrix}\psi(\mathbf{p}) = 0,$$

which when multiplied by m and upon identifying the matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

can be cast in conventional Dirac form

$$[\gamma^\mu p_\mu - \pi m]\psi(\mathbf{p}) = 0. \tag{32}$$

This shows that Dirac equation is just the parity (actually "intrinsic" parity, see below) eigenvalue equation as seen in a frame where the particle has a momentum $\mathbf{p}$. It can be shown that the whole structure of the Dirac theory arise naturally from here. In particular the projectors over "positive (negative) energy" subspaces are just projectors over well defined positive (negative) parity. States satisfying this condition for $\pi = 1$ describes particles while anti-particles correspond to the subspace with $\pi = -1$, this assignment being of course arbitrary. Finally, the γ matrices turn out to be conventional Dirac matrices in Weyl representation and satisfy the Clifford algebra which arise as a byproduct of the structure of the Poincaré group and the specific representation chosen.

We would like to emphasize that Dirac spinors in Eq.(30) are eigenstates of the full parity operation which we will denote as $\mathbb{P}$, composed of the "intrinsic" part, Π, and the operation changing $\mathbf{p} \to -\mathbf{p}$. Indeed

$$\mathbb{P}\psi(\mathbf{p}) \equiv \Pi\psi(-\mathbf{p}) = \Pi B^{-1}(\mathbf{p})\psi(\mathbf{0}) = B(\mathbf{p})\Pi\psi(\mathbf{0}) = \pi B(\mathbf{p})\psi(\mathbf{0}) = \pi\psi(\mathbf{p}).$$

The observation that Dirac equation is just the invariant form of the "intrinsic parity" (Π) eigenvalue equation in $(1/2, 0) \oplus (0, 1/2)$ space is by no means a trivial one since electromagnetic interactions of charged leptons are dictated by the $U(1)$ gauge principle in Dirac equation (modulo corrections due to the exchange of $W^\pm$ bosons). Hence, *the electromagnetic properties of interacting fermions are directly related to the discrete symmetry upon which the free particle description is based, in this case parity.* This observation becomes highly non trivial for theories in extra dimensions since the classification of discrete symmetries depends on the dimension of the space time as will be shown in the next section section.

4. Poincaré group and discrete symmetries in arbitrary dimensions

It is straightforward to show that the structure of this group does not depend on the number of space dimensions , i.e. $\mathbf{T}^{(1,n)}$ is an Abelian invariant subgroup of $\mathbf{P}(1,n)$ and this group is a semidirect product of $\mathbf{T}^{(1,n)}$ and the corresponding HLG which we denote as HLG_n.

$$\mathbf{P}(1,n) = \mathbf{T}^{(1,n)} \circledS HLG_n.$$

Furthermore, HLG_n can also be decomposed at the classical level in terms of four disjoint sets connected by discrete symmetries. One of these sets contains the identity and forms a subgroup, the group of proper and ortochronous Lorentz transformations in $d = 1 + n$ which we will denote as $L_+^{\uparrow}(n)$. However, in general, *the discrete symmetry connecting this piece to* $L_-^{\uparrow}(n)$ *is not the naive generalization of the usual space inversion* in $d = 1 + 3$. Indeed, notice that in space-time with an even number of space dimensions, conventional parity conceived as the inversion of space components, is a *proper* transformation, it has determinant $+1$, and as such it belongs to $L_+^{\uparrow}(n)$. In an odd dimension space-time (even number of space components) there exists several improper discrete transformations connecting $L_+^{\uparrow}(n)$ with $L_-^{\uparrow}(n)$. These transformations change the sign of an odd number of space coordinates and are all equivalent. For definitiveness we will choose the transformation changing the sign of the last space coordinate and will denote it as $\mathcal{M}$

$$\mathcal{M} = Diag(1, 1, ..., 1, -1). \tag{33}$$

Notice that in $d = 1 + 3$ this transformation amounts to a "mirror reflection" on the $x - y$ plane. In general, this is a mirror reflection on a given plane and for an even-dimensional space-time (odd number of space components), this transformation is equivalent to space inversion. We remark that this "mirror reflection" has been adopted in the literature as "parity". In order to avoid possible confusion in the nomenclature we name it "mirror reflection" here and the transformation reversing all the spatial coordinates is denoted as "spatial inversion".

Now we turn to the quantum case. We assume that quantum principles are independent of the dimension of space-time, i.e. systems are still described by states in the irreducible representations of their symmetry group. The Lie algebra of $L_+^{\uparrow}(n)$ is given by

$$\begin{aligned}
&[P^\mu,\, P^\rho] = 0,\\
&[J^{\rho\sigma},\, P^\mu] = i\left(\eta^{\sigma\mu}P^\rho - \eta^{\rho\mu}P^\sigma\right),\\
&[J^{\mu\nu},\, J^{\rho\sigma}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho}\right),
\end{aligned} \tag{34}$$

where now the indices run as $\mu = 0, 1, ..., n$, for the corresponding n. We implement discrete symmetries along with Eqs. (17) but now we must also consider "mirror reflection" defined as

$$M \equiv U(0, \mathcal{M}).$$

Again, considering infinitesimal transformations in Hilbert space we get

$$\begin{aligned}
\Pi\,(iJ^{\mu\nu})\,\Pi^{-1} &= i\mathcal{P}_\rho{}^\mu\mathcal{P}_\sigma{}^\nu J^{\rho\sigma},\\
\Pi\,(iP^\mu)\,\Pi^{-1} &= i\mathcal{P}_\rho{}^\mu P^\rho,\\
\Theta\,(iJ^{\mu\nu})\,\Theta^{-1} &= i\mathcal{T}_\rho{}^\mu\mathcal{T}_\sigma{}^\nu J^{\rho\sigma},\\
\Theta\,(iP^\mu)\,\Theta^{-1} &= i\mathcal{T}_\rho{}^\mu P^\rho,\\
M\,(iJ^{\mu\nu})\,M^{-1} &= i\mathcal{M}_\rho{}^\mu\mathcal{M}_\sigma{}^\nu J^{\rho\sigma},\\
M\,(iP^\mu)\,M^{-1} &= i\mathcal{M}_\rho{}^\mu P^\rho,
\end{aligned} \tag{35}$$

where $\mathcal{P}_\rho{}^\mu$, $\mathcal{T}_\rho{}^\mu$ and $\mathcal{M}_\rho{}^\mu$ are the representations for spatial inversion, time reversal and "mirror reflection" in the $d = 1 + n$-dimensional Minkowski space. It is straightforward to show that also in $d = 1 + n$ spatial inversion must be realized as a linear and unitary operation and time reversal as an anti-linear and anti-unitary one. As for "mirror reflection" it must be implemented as a linear and unitary transformation. Under this realization Eqs.(22) still hold and the transformation of the generators under "mirror reflection" reads

$$\begin{aligned}
&M\,\mathbf{P}\,M^{-1} = \widetilde{\mathbf{P}},, \qquad M\mathbf{K}\,M^{-1} = \widetilde{\mathbf{K}},\\
&M\,J^{ij}M^{-1} = \left\{\begin{array}{ll} J^{ij}, & \text{if } i,j \neq n\\ -J^{ij}, & \text{if } i = n \text{ or } j = n\end{array}\right\}
\end{aligned} \tag{36}$$

where the tilded vector is defined as $\widetilde{\mathbf{V}} = (V_1, ..., V_{n-1}, -V_n)$.

At this point we wonder about the nature of electrons in arbitrary dimensions from this perspective. *We consider electrons as states with well defined parity transforming in the lowest dimensional irreps of the proper orthochronous Lorentz subgroup (extended by parity if necessary).* In the next section we present the complete construction, from first principles, of these states in the simplest case $d = 1+2$, i.e. states describing planar electrons, using a similar procedure to the one leading to the states in Eq.(30) which describe electrons (and positrons) in $d = 1 + 3$.

5. Fermions in $d = 1 + 2$ from first principles

The group $L_+^{\uparrow}(2)$ is locally isomorphic to $SO(1, 2)$.

The calculation of the corresponding irreps can be found e.g. in [6] and we reproduce it in the appendix in order to fix our conventions. There is only one "angular momentum", $J \equiv J^{12}$ and two boost operators, $K^i \equiv J^{0i}$, $i = 1, 2$, in this case. Also we have only one Casimir operator, $C \equiv J^2 - K_1^2 - K_2^2$. The finite dimensional irreps of the $L_+^\uparrow(2)$ are labeled by the eigenvalues of the Casimir operator, $c(c+1)$, where $c \equiv n/2$ with n a nonnegative integer. These are subspaces of dimension $d = 2c+1$ spanned by the states $|c,m\rangle$, where m are the eigenvalues of J ("spin projection") taking the values $m = -c, -c+1 ..., c-1, c$.

The lowest dimensional irrep (beyond the trivial one) is obtained setting $c = 1/2$ in whose case we obtain $C = 3/4\mathbf{1}$. This irrep has dimension 2 and the states $\{|1/2, 1/2\rangle, |1/2, -1/2\rangle\}$ constitute a basis for this subspace. In this basis the operators J, J_+, J_- defined in the appendix have the following matrix representations

$$\begin{aligned} J &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ J_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{37}$$

whereas the states themselves have the following representation

$$\begin{aligned} \left|\frac{1}{2}, \frac{1}{2}\right\rangle &\to \psi^+(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle &\to \psi^-(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{38}$$

where the superindex denote the eigenvalue of J ("spin" projection). From Eq. (A.3) we obtain the following representations for the generators of $L_+^\uparrow(2)$

$$J = \frac{1}{2}\sigma_3, \qquad K_1 = \frac{i}{2}\sigma_1, \qquad K_2 = \frac{i}{2}\sigma_2. \tag{39}$$

The boost and rotation operators for this representation read

$$\begin{aligned} B(\mathbf{n}, \varphi) &= \cosh\frac{\varphi}{2} + \sigma\cdot\mathbf{n}\sinh\frac{\varphi}{2}, \\ D(\theta) &= \cos\frac{\theta}{2} - i\sigma_3\sin\frac{\theta}{2}, \end{aligned} \tag{40}$$

where $\mathbf{n} = (n_1, n_2)$ is the vector in the plane along the direction of the boost and $\sigma = (\sigma_1, \sigma_2)$. In the case of a boost acting on rest frame states the rapidity φ is related to p^μ through the analogous relations to Eq. (25), where now $\mathbf{p} = (p_1, p_2)$. In this case, the boost operator and its square to be used below can be written as

$$\begin{aligned} B(\mathbf{n}, \varphi) &= \frac{E + m + \sigma\cdot\mathbf{p}}{\sqrt{2m(E+m)}}, \\ B^2(\mathbf{n}, \varphi) &= \frac{E + \sigma\cdot\mathbf{p}}{m}. \end{aligned} \tag{41}$$

Acting with the boost operator and the rest frame states in the $\{|1/2, \pm1/2\rangle\}$ basis we obtain the following spinors

$$\begin{aligned} \psi^+(\mathbf{p}) &= \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} E+m \\ p_x + ip_y \end{pmatrix}, \\ \psi^-(\mathbf{p}) &= \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} p_x - ip_y \\ E+m \end{pmatrix}. \end{aligned} \tag{42}$$

These are the explicit form of the spinors, representing states in the lowest dimensional representation for a planar Minkowski world, constructed from first principles. In the next section we show that these are also eigenstates of space inversion thus they describe planar electrons.

6. Spatial inversion in $d = 1 + 2$

As discussed above, in sharp contrast with the physics in 1 + 3 dimensions, *spatial inversion transformation is not an improper transformation* in $d = 1 + 2$. At the Minkowski space level this transformation is

$$\begin{pmatrix} t' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} t \\ x \\ y \end{pmatrix}, \tag{43}$$

and the corresponding matrix $\mathcal{P} = \mathrm{Diag}\{1, -1 - 1\}$ satisfy $\mathrm{Det}(\mathcal{P}) = 1$. It is easy to see this transformation is just a rotation in an angle π.

The transformation properties for the generators of rotations and boosts in Hilbert space under spatial inversion can be inferred from Eq. (43) according to Eq. (35) as

$$\Pi J \Pi = J \qquad \Pi K_i \Pi = -K_i, \tag{44}$$

whereas for the Casimir and ladder operators we get

$$\Pi C \Pi = C, \qquad \Pi J_\pm \Pi = -J_\mp, \tag{45}$$

hence *the irreps $\{|c,m\rangle\}$ are also invariant subspace of spatial inversion.* This comes as not surprise since as we noticed above spatial inversion belongs to $L_+^\uparrow(2)$. From Eqs. (44),(45) it is straightforward to show that for a general irrep

$$\Pi|c,m\rangle = \eta(c,m)|c,m\rangle, \tag{46}$$

where $\eta(c,m)$ is a phase whose dependence on m can be fixed using the transformation properties of the ladder operators in Eq. (45) to $\eta(c,m) = (-1)^m\eta^{(c)}$ where $\eta^{(c)}$ is a global phase for a given irrep characterized by the quantum number c. For the specific case $c = 1/2$, we get the following representation for parity in the basis $\{|1/2, 1/2\rangle, |1/2, -1/2\rangle\}$

$$\Pi^{(1/2)} = i\eta^{(1/2)}\sigma_3. \tag{47}$$

Notice that modulo a phase this is the very same operator for a rotation in an angle π as expected. We adopt the convention $\eta^{(1/2)} = -i$ in the following.

6.1. Spatial inversion and Dirac Equation for $d = 1 + 2$

A general transformation in $L_+^{\uparrow}(2)$

$$U_\Lambda(\theta, \varphi) = e^{i(J\theta + \mathbf{K}\cdot\varphi)}, \tag{48}$$

satisfy

$$\Pi[U_\Lambda(\theta, \varphi)]^\dagger \Pi = [U_\Lambda(\theta, \varphi)]^{-1}, \tag{49}$$

which can be used to show that the product

$$(\psi, \phi) \equiv \langle\psi|\Pi|\phi\rangle. \tag{50}$$

is invariant under these transformations.

Now we consider the parity eigenvalue equation for spinors in the $\{|1/2, \pm 1/2\rangle\}$ basis and in the rest frame

$$\Pi\psi(0) = \pi\psi(0), \tag{51}$$

where $\pi = \pm 1$. Boosting this equation and using Eq. (49) for the case $\theta = 0$ (boosts) we get

$$\left[B^2(\mathbf{p})\Pi - \pi\right]\psi(\mathbf{p}) = 0. \tag{52}$$

which upon using Eqs.(41) can be rewritten to

$$\left[(E + \sigma\cdot\mathbf{p})\sigma^3 - \pi m\right]\psi(\mathbf{p}) = 0. \tag{53}$$

If now we define $\gamma^0 = \sigma^3$, $\gamma^1 = \sigma^3\sigma^1 = i\sigma^2$, $\gamma^2 = \sigma^3\sigma^2 = -i\sigma^1$ this equation can be written in standard Dirac form

$$[\gamma^\mu p_\mu - \pi m]\,\psi(\mathbf{p}) = 0. \tag{54}$$

Clearly, the solutions to this equation are the states in Eq.(42). This derivation shows that also in $d = 1 + 2$ Dirac equation is just the covariant form of "intrinsic" spatial inversion eigenvalue equation. The formal structure is similar to the case $d = 1 + 3$, the solutions to this equation are the spatial inversion eigenstates as seen in a frame where the particle has momentum $p^\mu = (E, \mathbf{p})$. Interestingly enough, under this perspective the mass term in the corresponding Lagrangian has the same transformation properties under parity and time reversal as in conventional Dirac theory in $d = 1 + 3$.

At this point it is necessary to remark that in spite of this formal analogy between the $d = 1 + 3$ and $d = 1 + 2$ cases, there are substantial differences in the physical content of Dirac equation. To start with, the non-relativistic limit of the states in the lowest dimensional irrep, $\psi^\pm(\mathbf{p})$ in Eq. (42)is

$$\begin{aligned} \psi^+(\mathbf{p}) &\longrightarrow \begin{pmatrix}1\\0\end{pmatrix} \\ \psi^-(\mathbf{p}) &\longrightarrow \begin{pmatrix}0\\1\end{pmatrix} \end{aligned} \tag{55}$$

a result which could lead us to identify these states as the conventional non-relativistic spin states of an electron. However, it is easy to show that states in Eq. (42) are eigenstates of spatial inversion

$$\mathbb{P}\psi^\pm(\mathbf{p}) \equiv \Pi\psi^\pm(-\mathbf{p}) = \pm\psi^\pm(\mathbf{p}), \tag{56}$$

which require to identify $\psi^+(\mathbf{p})$ as the state describing a particle and $\psi^-(\mathbf{p})$ as the anti-particle. Notice that this requires that a planar electron has fixed "spin" projection $m = +1/2$ and the planar positron has also a fixed "spin" projection $m = -1/2$. In order to further clarify this point next we construct charge conjugation operator in $d = 1 + 2$ and perform a non-relativistic expansion of Dirac equation. Before this, we remark that spinors describing planar electrons are not eigenstates of "mirror reflection" operation (adopted as parity in the current literature). Indeed, from Eqs. (35) it is possible to show that the matrix representation for a general irrep labeled by c is

$$(M^c)_{m'm} = \langle c, m'|M|cm\rangle = \xi^{(c)}\delta_{m'-m} \tag{57}$$

where $|\xi^{(c)}| = 1$. In particular for the spinor representation $c = 1/2$ we get

$$M = \xi^{(1/2)}\sigma^1. \tag{58}$$

Spinors in Eq. (42) are not eigenstates of this operator nor of its generalized form which include the change $\mathbf{p} \to \widetilde{\mathbf{p}}$.

7. Charge conjugation

Coupling to an external electromagnetic field via the gauge principle in configuration space we get

$$[(i\partial_\mu + eA_\mu)\gamma^\mu - m]\,\psi(x) = 0, \tag{59}$$

where e denotes the charge of the particle. We define charge conjugation as the operation transforming

$$C : \left\{ \begin{array}{ccc} e & \longrightarrow & -e, \\ \psi & \longrightarrow & \psi_c, \end{array} \right\}, \tag{60}$$

in such a way that the conjugate spinor ψ_c describes a particle with a charge $-e$ and thus satisfy

$$[(i\partial_\mu - eA_\mu)\gamma^\mu - m]\,\psi_c(x) = 0. \tag{61}$$

Complex conjugating Eq. (59) and applying $U_c\gamma^0$ on the left, with U_c a unitary matrix we obtain

$$\begin{aligned}\big[(i\partial_\mu - eA_\mu)(U_c\gamma^0)\gamma^{\mu *}(U_c\gamma^0)^{-1} + m\big]& \\ \times\,\psi_c(x) = 0,&\end{aligned} \tag{62}$$

where we defined the charge conjugate spinor as

$$\psi_c(x) = U_c\gamma^0\psi^*(x) = U_c\bar{\psi}^T, \tag{63}$$

with $\bar{\psi} = \psi^{\dagger}\gamma^{0}$. Eq. (61) requires

$$(U_c\gamma^0)\gamma^{\mu *}(U_c\gamma^0)^{-1} = -\gamma^{\mu}. \quad (64)$$

Since

$$\gamma^0\gamma^{\mu *}\gamma^0 = \gamma^0(\gamma^0, \gamma^1, -\gamma^2)\gamma^0 = (\gamma^0, -\gamma^1, \gamma^2),$$

Eq. (64) amounts to

$$U_c\gamma^1 U_c^{-1} = \gamma^1, \quad U_c\gamma^k U_c^{-1} = -\gamma^k \quad (k = 0, 2). \quad (65)$$

These relations are satisfied by

$$U_c = -\eta_c\sigma^2 = \eta_c\gamma^2\gamma^0, \quad (66)$$

where $|\eta_c|^2 = 1$. This operator has the following properties

$$\begin{aligned} U_c^T &= -U_c, \qquad U_c = -\gamma^0 U_c \gamma^0, \\ U_c^2 &= (\eta_c)^2 \mathbf{1}_2, \qquad \det(U_c) = -\eta_c^2. \end{aligned} \quad (67)$$

Finally the charge conjugate spinor is

$$\psi_c(x) = U_c\bar{\psi}^T = \eta_c\gamma^2\psi^*(x) = i\gamma^2\psi^*(x). \quad (68)$$

Here we have choosen the phase convention $\eta_c = i$ in such a way that

$$(\psi^+)_c = \psi^-, \qquad (\psi^-)_c = \psi^+, \quad (69)$$

hence ψ^+ describes a particle of positive spatial inversion and ψ^- is indeed its anti-particle with the opposite spatial inversion. This requires particles and antiparticles to have fixed "spin" projection as mentioned above. That this is the case can also be shown by taking the non-relativistic limit of Dirac equation. Considering the weak external field limit and writing the Dirac field as

$$\psi = \begin{pmatrix} \widetilde{\varphi}_u \\ \widetilde{\varphi}_d \end{pmatrix} = e^{-i\left(mc^2/\hbar\right)t} \begin{pmatrix} \varphi_u \\ \varphi_d \end{pmatrix} \equiv e^{-i\left(mc^2/\hbar\right)t}\phi, \quad (70)$$

where we re-introduced the $\hbar$ and c factors, a straightforward calculation yields

$$\varphi_d = \frac{p_x + ip_y}{2mc}\varphi_u, \quad (71)$$

and the "large" component satisfy

$$i\hbar\frac{\partial\varphi_u}{\partial t} = \left(\frac{\pi^2}{2m} + \frac{e\hbar}{2mc}B - e\Phi\right)\varphi_u \quad (72)$$

where $B \equiv \nabla \times \mathbf{A} = \epsilon_{ij}\,\partial_i A_j = \partial_1 A_2 - \partial_2 A_1$. This is the analogous in $d = 1 + 2$ of the Pauli equation and shows that in a planar world the Dirac particle has a fixed "spin" projection $m = 1/2$. This is in agreement with a calculation of irreps of the Poincaré group in $d = 1 + 2$ using the little group techniques in Ref. [7]

8. Summary

In this work we calculate spinors describing particles transforming in the lowest dimensional irrep of the Poincaré group in $d = 1 + 2$ from first principles (without any reference to equations of motion or Lagrangians). We show that these spinors are eigenstates of conventional parity (space inversion) and satisfy Dirac equation which in this perspective turns out to be just the covariant form of the eigenvalue equation for intrinsic parity, just like in $d = 1 + 3$. The construction of charge conjugation operator and the non-relativistic expansion of the Dirac equation shows that, in a planar world, electrons have fixed "spin" projection $m = +1/2$ while planar positrons have always the opposite "spin" projection $m = -1/2$.

Appendix A. Irreducible representations of $\mathbf{L}_+^{\uparrow}(2)$

The commutation relations for the generators of rotations, $J \equiv J^{12}$ and two boost operators, $K^i \equiv J^{0i}$, $i = 1, 2$ are extracted from Eqs. (34) taking $n = 2$ as

$$\begin{aligned} [J, K_1] &= iK_2, \\ [J, K_2] &= -iK_1, \\ [K_1, K_2] &= -iJ. \end{aligned} \quad (A.1)$$

The Casimir operator is $C = J_3^2 - K_1^2 - K_2^2$. This can be directly verified or inferred from the mapping $J_1 \to iK_1, J_2 \to iK_2$ which transforms the above algebra into the $SU(2)$ one. Now we choose J as a second operator to label the states and denote the $\{C, J_3\}$ eigenstates as

$$C|a, b\rangle = a|a, b\rangle, \qquad J|a, b\rangle = b|a, b\rangle. \quad (A.2)$$

Next, we define the "ladder" operators as[1]

$$J_{\pm} \equiv -iK_1 \pm K_2 \quad (A.3)$$

which satisfy the commutation relations

$$[J_+, J_-] = 2J, \quad [J, J_{\pm}] = \pm J_{\pm}, \quad [C, J_{\pm}] = 0, \quad (A.4)$$

also

$$J_- J_+ = C - J^2 - J, \quad J_+ J_- = C - J^2 + J. \quad (A.5)$$

Using relations (A.4) we obtain

$$JJ_{\pm}|a, b\rangle = (b \pm 1)(J_{\pm}|a, b\rangle), \quad (A.6)$$

and

$$CJ_{\pm}|a, b\rangle = aJ_{\pm}|a, b\rangle, \quad (A.7)$$

[1] This convention differ from the one used in [6]. Our convention give a simpler form for the generators of boosts in Eq. (39).

hence $J_\pm\ |a,b\rangle \sim |a,b \pm 1\rangle$. Using this repeatedly we obtain $(J_\pm)^n|a,b\rangle \approx |a,b+n\rangle$ where n is a non-negative integer. Now, from Eqs. (A.5) we obtain

$$C - J^2 = \frac{1}{2}(J_+J_- + J_-J_+). \tag{A.8}$$

Here, we have two possibilities: the first one is to consider K_1,K_2 hermitian (hence boost will be unitary). In this case we obtain infinite dimensional representations [6] we are not interested in here. The second possibility is to choose K_1,K_2 not hermitian (hence boost will be non-unitary). We consider the possibility realizing

$$J_- = (J_+)^\dagger, \tag{A.9}$$

in whose case

$$K_i^\dagger = -K_i. \tag{A.10}$$

The calculation now goes along the line of the conventional calculation of the irreps of $SU(2)$. As a result we obtain

$$a = c(c+1), \qquad b = m, \tag{A.11}$$

where $c = n/2$, with n a non-negative integer and $m = -c, -c+1, ..., c-1, c$. In this notation

$$\begin{aligned} C|cm\rangle &= c(c+1)|cm\rangle, \\ J|cm\rangle &= m|cm\rangle, \\ J_\pm|cm\rangle &= r^\pm_{cm}|c,m\pm 1\rangle, \end{aligned} \tag{A.12}$$

where we choose the usual phase conventions making $r^\pm_{cm}$ real and given as

$$r^\pm_{cm} = \sqrt{(c \mp m)(c \pm m + 1)}. \tag{A.13}$$

Summarizing, the finite dimensional irreps of the $L^\uparrow_+(2)$ are labeled by $c \equiv n/2$ with n a nonnegative integer and are subspaces of dimension $d = 2c+1$.

The explicit matrix representation of J and the ladder operators is

$$\begin{aligned} \langle|J|cm\rangle \quad &= m\delta_{\acute{m}m}\ , \\ \langle cm\!\!\restriction\! J_+|cm\rangle &= r^+_{cm}\delta_{\acute{m},m+1}, \\ \langle cm\!\!\restriction\! J_-|cm\rangle &= r^-_{cm}\delta_{\acute{m},m-1}. \end{aligned} \tag{A.14}$$

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