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Conformal Transformation and Jacobi Matrix

Contents

Abstract

In this paper we study a class of conformal transformations defined on *comb* type regions. We estimate some geometric properties of these particular regions and also we study the relationship between this particular class of conformal transformations and the Jacobi matrix.

1. Introduction

Let us denote by $\mathbb{C}_\pm = \{z = x + iy \in \mathbb{C} : \pm\text{Im } z > 0\}$ the complex upper and lower half plane according to the sign. We want to look at the region $K_+(h) = \mathbb{C}_+ \setminus \bigcup_{j=1}^{n-1} \Gamma_j$, where n is a fixed positive number, and Γ_j is a hyperbolic slit given by the formulae: $u = \sin s_j \text{sh } \eta$, whose focci are in $k = \pm 1$, where $k = u+iv \in K_+(h)$ y $s_j = (2j-n)/(2n)$, $\eta = t/n$, $h = \{h_j\}_{j=1}^{n-1}, 0 \le h_j < \infty,\ 0 \le t \le h_j,\ j = 1, \dots, n-1$.

In this work we study a particular class of functions $z(k)$, giving a conformal transformation of the region $K_+(h)$ on $\mathbb{C}_+$ which represents an asymptotic expanssion $z(k) = k + Q_0/k + o(1/k),\ k \to \infty$.

According to Hilbert's theorem [4] about the conformal transformations of multiply connected regions on regions with paralell slits, there exists a unique analytic function that transforms conformally K_+ in $\mathbb{C}_+$ and such that it has an asymptotic expanssion $z(k) = k + Q_0/k + o(1/k)$ at infinity. Furthermore $z(k)$ is characterized by the property that among the functions defined on K_+ having asymptotic expanssion $f(k) = k + a_1/k + o(1/k),\ k \to \infty$, it is satisfied Re $a_1 <$ Re Q_0.

In this work we find an expression for the coefficient Q_0 in terms of the entries of certain periodical Jacobi matrix.

The study of the conformal transformations of the complex half plane over regions where there have been removed a finite or infinite ammount of continuous of a well defined form (call them slits) is an active field of research. When the slits are vertical, they are generally known as *comb* type regions. We can say that our region $K_+(h)$ is a particular type of *comb* region, and we will refer to it like this. In [1], B.Ya. Levin studied the general properties of the conformal transformations defined on *comb* type regions Ω of arbitrary form; he stablished, among other things that the finite subharmonic majorant of certain class of functions coincides with the imaginary part of a function which transforms in a conformal way $\mathbb{C}+$ in Ω and also proved the converse.

We have to point out that in several branches of mathematics the transformations defined in comb spaces of a given form are used; soem examples are: the Löwner equation theory in plane electrostatic, some problems related to the concept of analytic capacity, in the spectral theory of operators with periodic coefficients, in problems related to Hill and Dirac operators, in Korteweg de Vries equation theory, and in the non linear Schroedinger equation with periodic initial conditions [4].

The conformal transformations on comb regions of defined form and their relationship with the spectral problem have been studied with some of their properties, in the papers by V.A. Marchenko [2];

V.A. Marchenko and I.V. Ostrovsky [3]; P.P. Kargaev and E.L. Korotiaev [4,5,6] and L.V. Perkolab [7]. The difference between these works and this one is that here we study hyperblolic slits (defined above) and not only vertical ones.

2. The Region $K_+(h)$

Let us define the region $K_+(h)$ as $K_+(h) = \mathbb{C}_+ \setminus \bigcup_{j=1}^{n-1} \Gamma_j$, where n is a fixed positive integer number, and each Γ_j is a hyperbolic slit, given by the formula: $u = \sin s_j \mathrm{sh}\ \eta$, with focci $k = \pm 1$, where $k = u + iv \in K_+(h)$ y $s_j = (2j-n)/(2n)$, $\eta = t/n$, $h = \{h_j\}_{j=1}^{n-1}, 0 \le h_j < \infty,\ 0 \le t \le h_j,\ j = 1, \ldots, n-1$.

Along our discussion it will be useful the theorem by Hilbert on the conformal transformations from a multiply connected region to a region with parallel slits. Let $S_1, S_2, \ldots, S_N$ be disjoint continua in the plane $\mathbb{C}$ and let us define $D = \mathbb{C} \setminus \bigcup_{n=1}^{N} S_n$. We introduce the class $\sum'(D)$ of conformal transformations w that send D onto $\mathbb{C}$ with the asymptotic $w(k) = k + [Q(w) + o(1)]/k,\ k \to \infty$. The following result is the well known Hilbert's theorem.[1]

Theorem 1. (Hilbert [4].) *Let $S_1, S_2, \ldots, S_N$ be a collection of disjoint continua in $\mathbb{C}$; $D = \mathbb{C} \setminus \bigcup_{n=1}^{N} S_n$. Then for each $t \in [0, \pi]$ there exists a unique map $w_t \in \sum'(D)$ sending D onto a region with parallel slits, whose intersection angles with the real axis is t. Furthermore for each function $f \in \sum'(D),\ f \ne w_0$, the following inequality holds* $\mathrm{Re}\ Q(f) < \mathrm{Re}\ Q(w_0)$.

Based on this theorem we assert that under certain symmetry conditions, there exists a function that gives a conformal transformation from $K_+(h)$ onto $\mathbb{C}+$, with asymptotic $z(k) = k + Q_0/k + o\,(1/k)\ ,\ k \to \infty$ and eith the extremal property from the theorem.

Let $\mathbb{C}$ be the complex plane. If we denote by $K(h) = \mathbb{C} \setminus \bigcup_{j=1}^{n-1} \Gamma_j$, where n is a fixed positive integer number, each Γ_j is a hyperbolic slit given by the formula: $u = \sin s_j \mathrm{sh}\ \eta$, with focci in $k = \pm 1$, where $k = u + iv \in K_+(h)$ and $s_j = (2j-n)/(2n)$, $\eta = t/n$, $h = \{h_j\}_{j=1}^{n-1}, 0 \le h_j < \infty$, $|t| \le h_j$, $j = 1, \ldots, n-1$. The the function $k : \mathbb{C}_+ \to K_+(h)$, by the symmetry principle may be continued to $\mathbb{C}$, by means of $\overline{k(\overline{z})} = k(z)$.

It happens that $z(k)$ may be represented as a composition of maps, to see this let us introduce a special class of conformal transformations of $\mathbb{C}_+$ in a vertical half band with $n-1$ vertical slits. This regions

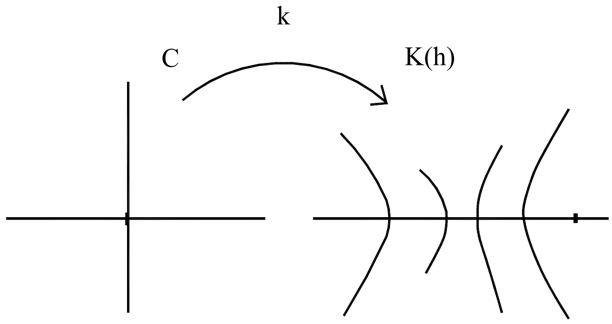

Fig. 1. The conformal transformation $k : \mathbb{C} \to K(h)$.

will be denoted by $D_n(h)$. Let $P_n(z)$ be a polynomial of degree n. Based on the results by V.A. Marchenko and I.V. Ostrovsky in [3] and L.V. Perkolab in [7], we assert that $P_n^2(z) = 1$ has only real roots if and only if this polynomial of degree n can be represented by the formula $P_n(z) = \cos\theta(z)$, where $\theta(z)$ is a conformal transformation that sends $\mathbb{C}_+$ to $D_n(h)$:

$$D_n(h) = \Big\{\theta : 0 < \mathrm{Re}\,\theta < n\pi,\ \mathrm{Im}\,\theta > 0\}\setminus \bigcup_{j=1}^{n-1} \{\theta : \mathrm{Re}\,\theta = j\pi,\ 0 \le \mathrm{Im}\,\theta \le h_j\Big\},$$

where $h = \{h_j\}_{j=1}^{n-1},\ 0 \le h_j < \infty$ and for $|z| \gg 1$ one has $\theta(z) \sim in \ln z$.

Let us apply the Laguerre's theorem [8] to the maps $P_n(z) + 1$ and $P_n(z) - 1$, so that the roots of the equations $P_n^2(z) = 1$ and $P_n'(z) = 0$ may be enumerated as follows: $a_0^- < a_1^+ \le \gamma_1 \le a_2^+ \le \gamma_2 \le a_2^- < a_3^+ \le \gamma_3 \le a_3^- < \cdots < a_{n-1}^- < a_n^+$, where $P_n(a_0^-) = +1$, $P_n(a_n^+) = -1$, $P_n(a_{2j}^+) = P_n(a_{2j}^-) = 1$, $P_n(a_{2j-1}^+) = P(a_{2j-1}^-) = -1$, $(j = 1, 2, \ldots, (n-1)/2)$ y $P_n'(\gamma_j) = 0$, $(j = 1, 2, \ldots, n-1)$.

Therefore, $\theta(z)$ which is a conformal transformation of $\mathbb{C}_+$ in $D_n(h)$, such that when $|z| \gg 1$, then $\theta(z) \sim in \ln z$ and sends the points a_0^- and a_n^+ to 0 and $n\pi$ respectively, may be expressed by the Schwarz-Christoffel formula

$$\theta(z) = ci \int_{\alpha_0}^{z} \prod_{k=1}^{n-1} (\xi - \gamma_k)\ d\xi \Bigg/ \sqrt{(\xi - \alpha_0) \left(\prod_{k=1}^{n-1} (\xi - \alpha_k)(\xi - \beta_k) \right) (\xi - \beta_n)},$$

where a_j^+, a_j^- are the preimages of the points $j\pi$, and γ_j are the preimages of $j\pi + ih_j$ $(j = 1, 2, \ldots, n-1)$ belonging to the region $D_n(h)$.

Let us define the function $\zeta(k) = \arcsin k$ as the

[1] The complete proof of this result can be found in: **Golusin, G.**, *Geometric theory of funtions of a complex variable.* Amr. Math. Soc., Providence, RI, 1969; y **Jenkins, A.**, *Univalent functions and conformal mapping.* Berlin, Göttingen, Heidelberg, Springer, 1958.

conformal transformation of $K_+(h)$ onto the region

$$\Pi_+ = \{\zeta : -\pi/2 < \operatorname{Re}\zeta < \pi/2,\ \operatorname{Im}\zeta > 0\}\backslash$$
$$\backslash \bigcup_{j=1}^{n-1} \{\zeta : \operatorname{Re}\zeta = (2j-n)\pi/2n,\ 0 \le \operatorname{Im}\zeta \le h_j/n\}$$

where $h_j < \infty$, with thw normalization $\zeta(\pm 1) = \pm\pi/2$. Thus we can write

$$z(k)\theta^{-1}[n(\zeta(k)+\pi/2)] = \theta^{-1}[n(\arcsin k + \pi/2)].$$

3. Results

Let us define the Dirichlet integral I_D for the transformation $z(k)-k$ as

$$I_D = \frac{1}{\pi}\iint\limits_{K_+(h)} |z'(k)-1|^2\,dudv = \frac{1}{\pi}\iint\limits_{\mathbb{C}_+} |k'(z)-1|^2\,dxdy,$$

where $k(z)$ is the inverse analytical function $z(k)$. This inequality holds due to the fact that the Dirichlet integral is invariant under conformal transformations.
Theorem 2. *Let $z(k)$ be a conformal transformation sending $K_+(h)$ to $\mathbb{C}_+$ and whose asymptotic is $z(k) = k+Q_0/k+o(1/k),\ k\to\infty$. Then the following equality holds*

$$I_D = \frac{1}{\pi}\iint\limits_{K_+(h)} |z'(k)-1|^2\,dudv$$
$$= \frac{1}{\pi}\iint\limits_{\mathbb{C}_+} |k'(z)-1|^2\,dxdy == \frac{1}{\pi}\int\limits_{\mathbb{R}} v(t)\,dtQ_0,$$

where $v(z) = \operatorname{Im}\,k(z),\ k(z)$ is an inverse function of $z(k)$.

Proof. We will use the methods developed in [5]. Let us take the map $k(z)-z$ written in the form $k(z)-z = R+iJ$, where $R(x) = u(x)-x,\ J(x) = v(x),\ x\in\mathbb{R}$. Thus we have $k(x)-x = R(x)+iJ(x)$, where $R(x) = u(x)-x$, y $J(x) = v(x),\ x\in\mathbb{R}$, y $u(z) = \operatorname{Re}\,k(z)$ y $v(z) = \operatorname{Im}\,k(z)$.

Now let us examine the region

$$\mathbf{D} = \{z : r_1 < |z| < r_2,\ y > 0\},\quad 0 < r_1 < r_2 < \infty$$

and the Dirichlet integral

$$I_D = \frac{1}{\pi}\iint\limits_{\mathbf{D}} |k'(z)-1|^2\,dxdy.$$

After certain transformations we will apply Green's formula for our case in $\mathbf{D}$

$$\pi I_D = \iint\limits_{\mathbf{D}} |k'(z)-1|^2\,dxdy$$
$$= \iint\limits_{\mathbf{D}} |(k'(z)-z)'|^2\,dxdy$$
$$= \iint\limits_{\mathbf{D}} \left[\left(\frac{\partial R}{\partial x}\right)^2 + \left(\frac{\partial J}{\partial x}\right)^2\right]\,dxdy.$$

Since $k(z)-z$ is analytic, we have

$$\iint\limits_{\mathbf{D}} |k'(z)-1|^2\,dxdy$$
$$\times \iint\limits_{\mathbf{D}} \left(\frac{\partial R}{\partial x}\frac{\partial J}{\partial y} - \frac{\partial R}{\partial y}\frac{\partial}{\partial x}\right)\,dxdy$$
$$= \iint\limits_{\mathbf{D}} \left[\frac{\partial}{\partial y}\left(J\frac{R}{\partial x}\right) - \frac{\partial}{\partial x}\left(J\frac{R}{\partial y}\right)\right]\,dxdy.$$

Let us apply now the Green's formula:

$$\pi I_D = \iint\limits_{\mathbf{D}} |k'(z)-1|^2\,dxdy$$
$$= -\iint\limits_{\mathbf{D}} \left[\frac{\partial}{\partial x}\left(J\frac{\partial R}{\partial y}\right) - \frac{\partial}{\partial y}\left(J\frac{\partial R}{\partial x}\right)\right]\,dxdy$$
$$= -\int\limits_{\partial\mathbf{D}} J\frac{\partial R}{\partial x}\,dx + J\frac{\partial R}{\partial y}\,dy,$$

where $\partial\mathbf{D}$ is the boundary of $\mathbf{D}$.

Let us divide $\partial\mathbf{D}$ in three parts: the real part $r_1 < |x| < r_2$, the semicircles $c_{r_2} = \{z : |z| = r_2, y > 0\}$ y $c_{r_1} = \{z : |z| = r_2, y > 0\}$. Let us consider also $x = t\cos\varphi,\ y = t\sin\varphi,\ r_1 \le t \le r_2,\ 0 < \varphi < \pi$, thus we obtain

$$\pi I_D = -\int\limits_{r_1<|x|<r_2} J\frac{\partial R}{\partial x}\,dx$$
$$+ \int_0^\pi J(r_1 e^{i\varphi})\frac{\partial R(r_1 e^{i\varphi})}{\partial\varphi}\,d\varphi$$
$$- \int_0^\pi J(r_2 e^{i\varphi})\frac{\partial R(r_2 e^{i\varphi})}{\partial\varphi}\,d\varphi.$$

Due to the analyticity of $k(z)-z$, we have

$$\pi I_D = -\int\limits_{r_1<|x|<r_2} J\frac{\partial R}{\partial x}\,dx$$
$$+ r_2\int_0^\pi J(r_2 e^{i\varphi})\frac{\partial J(r_2 e^{i\varphi})}{\partial t}\,d\varphi$$
$$- r_1\int_0^\pi J(r_1 e^{i\varphi})\frac{\partial J(r_1 e^{i\ varphi})}{\partial t}\,d\varphi.$$

We write this expression in its equivalent form

$$\pi I_D = -\int\limits_{r_1<|x|<r_2} R'(x)J(x)\ dx + \frac{r_2 b'(r_2) - r_1 b'(r_1)}{2},$$

where

$$b(t) = \int\limits_0^\pi J^2(te^{i\varphi})\ d\varphi.$$

The left hand side summand has the form

$$-\int\limits_{r_1<|x|<r_2} R'(t)J(t)\ dt$$

$$= \int\limits_{r_1<|x|<r_2} v(t)\ dt - \int\limits_{r_1<|x|<r_2} v(t)du(t),$$

since $-R'(x)J(x) = v(x)(1-u'(x)),\ x \in \mathbb{R}$.

If we consider that when one runs over a part of the boundary of $\mathbb{C}_+$, $v(x) = 0$, along other part of the same boundary, $v(x)$ runs over each hyperbolic slit in opposite direction, so that,

$$\int\limits_{r_1<|x|<r_2} v(t)\ du(t)$$

is zero when $r_1 \to 0,\ r_2 \to \infty$

Let us stretch out $|b'(t)|$, as $t \to \infty$.

>From the fact that $k(z) - z = R(z) + iJ(z),\ z \in \mathbb{C}_+$, $|J(z)| < |k(z) - z|$ y $k(z) = z - Q_0/z + o(1/z),\ z \to \infty$, we have

$$|b'(t)| \sim \frac{2cQ_0^2}{t^3},\ t \to \infty,\ c \text{ is a constant.}$$

Therefore,

$$|b'(t)| \to 0, \quad t \to \infty.$$

Finally, by the Theorem of Hamburger-Nevalinna [9], we obtain

$$\frac{1}{\pi}\int\limits_{\mathbb{R}} v(x)\ dxQ_0$$

and furthermore

$$I_D = \frac{1}{\pi}\iint\limits_{\mathbb{C}_+} |k'(z) - 1|^2\ dxdy = Q_0$$

so the theorem in proved. ■

A central issue for the geometric estimation of $K_+(h)$ is the following lemma, which is a variant of the Phagmen-Lindelöf principle.

Lemma 3. *Let $z_1(k)$ y $z_2(k)$ be two conformal transformations that send W_1 and W_2, respectively in $\mathbb{C}_+$. $U(A)\bigcap\{k \in \mathbb{C} : |k| > R\} \subset W_2 \subset W_1$, where $U(A) = \{k = u + iv : v > A|u|\}$ y $z_m(k) = k + o(k),\ k \to \infty,\ k \in U(A),\ m = 1, 2$. Then* $\text{Im } z_2(k) \le \text{Im } z_1(k),\ k \in W_2$.

If we assume that $H^+ = \max_{1\le j\le n-1}\{H_j\}$, where $H_j = \cos(2j-n)/(2n)\text{sh }(h_j)/n,\ 0 \le h_j < \infty,\ j = 1, 2, \ldots, n-1$, then the following result holds.

Theorem 4. *Let $z(k)$ be a conformal transformation of $K_+(h)$ onto $\mathbb{C}_+$ having an asymptotic expanssion $z(k) = k + Q_0/k + o(1/k),\ k \to \infty$. Then*

$$H^+ \le \pi\sqrt{6Q_0}.$$

Proof. Let the Dirichlet integral be

$$\iint\limits_{K_+(h)} |z'(k) - 1|^2\ dudv,$$

where $z : K_+(h) \to \mathbb{C}_+$, with asymptotic expanssion $z(k) = k + Q_0/k + o(1/k),\ k \to \infty$.

Making $\psi(k) = v - y(k),\ \text{Im } z(k) = y(k),\ k = u = iv \in K_+(h)$, we obtain

$$\iint\limits_{K_+(h)} |z'(k) - 1|^2\ dudv = \iint\limits_{K_+(h)} |\nabla\psi|^2\ dudv.$$

Let $k : \mathbb{C}_+ \to K_+(h)$ y $z_1(k)$ the inverse function of $k(z)$. Let us write $K'_+(h)$ to denote the region $\mathbb{C}_+ \bigcap K'(h)$, wheree $K'(h) = \mathbb{C}\backslash\Gamma_p$, Γ_p is a hyperbolic slit given by $u_p = \sin(2p-n)\pi/\text{ ch }(t/n),\ v_p = \cos(2p-n)\pi/2n\text{sh }(t/n)$, with focci $k = \pm 1$, $k = u+iv \in K'(h)$, $0 \le t \le h_p,\ 0 \le h_p < \infty$.

Let $z_2 : K'_+(h) \to \mathbb{C}_+$. We inscribe the slit Γ_p inside a square D whose side is $H^+ = v_p = \cos(2p-n)/(2n)\pi\text{ sh }(h_p)/n$. Así, $D[a - H^+, a + H^+] \times [0, H^+]$, where $a = \sin(2p-n)\pi/2n$.

Let us consider $\rho \in (H^+/\sqrt{2}, H^+),\ \pi/4 < \varphi(\rho) < 3\pi/4$. After that let us take the contour $c_p = \{k \in K'_+(h) : |k - a| = \rho,\ 0 \le \arg k \le \varphi(\rho)\}$ with clockwise orientation, and being $\varphi(\rho)$ the intersection angle between the contour and Γ_p.

Let us take $\psi(k) = v - y(k)$ which is harmonic in $K'_+(h)$, where $y(k) = \text{Im } z_2$. Let us poin out that $\psi(k) = v$ when $k \in \partial K'_+(h)$. $\partial K'_+(h)$ is the boundary of $K'_+(h)$.

Thus we have

$$\frac{\rho}{\sqrt{2}} \le \sin\varphi(\rho) \le \rho = \left|\int\limits_{c_p} d\psi\right| \le \int\limits_{c_p}\left|\frac{\partial\psi}{\partial u}\ du + \frac{\partial\psi}{\partial v}\ dv\right|.$$

Using the concept of total derivative of the function ψ and

$$\nabla\psi = \frac{\partial\psi}{\partial u} + i\frac{\partial\psi}{\partial v},$$

we obtain

$$\frac{\rho}{\sqrt{2}} \le \int\limits_{c_p} |\nabla\psi||dk| = \rho\int\limits_0^{\varphi(\rho)} |\nabla\psi(\rho e^{i\vartheta})|\ d\vartheta.$$

Using the Cauchy-Buniakovsky inequality, multiplying by $\rho\ d\rho$ and the integratin from $H^+/\sqrt{2}$ to H^+, we have

$$\int_{H^+/\sqrt{2}}^{H_+} \rho\ d\rho \le \frac{3}{2}\pi\rho \int_{H^+/\sqrt{2}}^{H^+} \int_0^{\varphi(\rho)} |\nabla\psi|^2\ d\vartheta d\rho$$
$$= \frac{3}{2}\pi \iint_{D^*} |\nabla\psi|^2\ dudv,$$

where $D^* = \{k \in K'_+(h) : H^+/\sqrt{2} \le |k-a| \le H^+,\ 0 \le \arg k \le \varphi(\rho)\},\ a = \sin(2p-n)\pi/2n$.

Therefore

$$(H^+)^2 \le 6\pi \iint_{D^*} |\nabla\psi|^2\ dudv,$$

that is

$$(H^+)^2 \le 6\pi \iint_{D^*} |z_2'(k)-1|^2\ dudv.$$

>From theorem 1 we obtain

$$(H^+)^2 \le 6\pi \iint_{D^*} |z_2'(k)-1|^2\ dudv \le 6\pi Q_0(z_2').$$

Considering that $K_+(h) \subset K'_+(h)$ and also the asymptotic expanssion of the functions z_1 and z_2, then by the above lemma we have

$$(H^+)^2 \le 6\pi \iint_{D^*} |z_2'(k)-1|^2\ dudv$$
$$\le 6\pi \iint_{K_+(h)} |z_1(k)-1|^2\ dudv.$$

Finally, by theorem 1 we obtain

$$H^+ \le \pi\sqrt{6Q_0}$$

and the proof is complete. ■

By fixing j, we can get an estimate for H_J in terms of Q_0.

Let $\varphi(x) = \mathrm{sh}\,(2x)/4 - x/2$, then it is clear that for certain absolute constants $C_1, C_2 > 0$ the following estimate holds

$$C_1(x^{1/2}+x^{1/3}) \le \varphi(x) \le C_2(x^{1/2}+x^{1/3}),\ x \in (0,+\infty).$$

Theorem 5. *Let j fixed, $j = p$, then for H_p holds the estimate*

$$H_p \le C[(\pi c_p Q_0)^{1/2} + (\pi c_p Q_0)^{1/3}],$$

where $C = C(C_1, C_2),\ c_p = (n-p)/n$.

Proof. We already know that $\theta(z)$ sends $\mathbb{C}_+$ onto the region $D_n(h)$:

$$D_n(h) = \{\theta : 0 < \mathrm{Re}\ \theta < n\pi,\ \mathrm{Im}\ \theta > 0\}$$
$$\setminus \bigcup_{j=1}^{n-1} \{\theta : \mathrm{Re}\ \theta = j\pi,\ 0 \le \mathrm{Im}\ \theta \le h_j\},$$

where $0 \le h_j \le \infty$ и $\theta(z) \sim in\ln z,\ |z| \gg 1$. We also have introduced the function $\zeta(k) = \arcsin k$, that sends the region $K_+(h)$ onto

$$\Pi_+ = \{\zeta : -\pi/2 < \mathrm{Re}\ \zeta < \pi/2,\ \mathrm{Im}\ \zeta > 0\}\setminus$$
$$\setminus \bigcup_{j=1}^{n-1} \{\zeta : \mathrm{Re}\ \zeta = (2j-n)\pi/2n,\ 0 \le \mathrm{Im}\ \zeta \le h_j/n,\}$$

wheree $h_j < \infty$, subject to the condition $\zeta(\pm 1) = \pm\pi/2$.

Let us assume that $k(\zeta)$ is an analytic function which is inverse of $\zeta(k)$ and that the function $\lambda(\zeta) = \theta^{-1}(n(\zeta + \pi/2)),\ \zeta = s + i\eta$- defined on the region Π_+, transforming it conformally in $\mathbb{C}_+$. Therefore $z(k(\zeta)) = \lambda(\zeta)$, where $z(k)$ is an analytic function transforming conformally $K_+(h)$ in $\mathbb{C}_+$, so that

$$\iint_{K_+(h)} |z'(k)-1|^2\, dudv$$
$$= \iint_{\Pi_+} |z'(k(\zeta))-1|^2 |k'(\zeta)|^2\, dsd\eta$$
$$= \iint_{\Pi_+} |z'(k(\zeta))k'(\zeta) - k'(\zeta)|^2\, dsd\eta$$
$$= \iint_{\Pi_+} |(z(k(\zeta)) - k(\zeta))'|^2\, dsd\eta$$
$$\times \iint_{\Pi_+} |(\lambda(\zeta) - \sin\zeta)'|^2\, dsd\eta.$$

Thus by theorem 1, we have

$$Q_0 = \frac{1}{\pi} \iint_{K_+(h)} |z'(k)-1|^2\, dudv$$
$$\times \frac{1}{\pi} \iint_{\Pi_+} |(\lambda(\zeta) - \sin\zeta)'|^2\, dsd\eta, \qquad \zeta = s + i\eta.$$

Let us take in the region Π_+ any segment:

$$\{\zeta : \mathrm{Re}\ \zeta = s_p,\ 0 \le \mathrm{Im}\ \zeta \le h_p/n\}, \qquad (3.11)$$

where $s_p = (2p-n)\pi/2n$. Let $0 \le s_p < \pi/2$ [2] and $c_p = \pi/2 - s_p = (n-p)\pi/n$.

[2] For the case $-\pi/2 < s_p(2j-n)\pi/2n \le 0$, the proof is analogous.

Let $\psi(\zeta) = \text{Im } (\sin\zeta - \lambda(\zeta))v(\zeta) - y(\zeta)$, $k = u + iv \in K_+(h), z = x + y \in \mathbb{C}_+$. Let us note that, if $\zeta \in \partial\Pi_+$, where $\partial\Pi_+$ is the boundary of Π_+, then $\text{Im } (\sin\zeta - \lambda(\zeta)) = v(\zeta)$. In the given segment we have

$$\begin{aligned}\text{Im } (\sin(s_p + i\eta) - \lambda(s_p + i\eta)) = v(\eta)\\ = \cos s_p \text{sh } \eta, \quad 0 \le \eta \le h_p/n, \ h_p < \infty.\end{aligned}$$

Therefore

$$\cos^2 s_p \int_0^{h_p/n} \text{sh}^2\ \eta\ d\eta \int_0^{h_p/n} v^2(\eta)\ d\eta.$$

Considering that $\psi(\zeta) = v(\zeta) - y(\zeta)$, we can write

$$\begin{aligned}\cos^2 s_p \int_0^{h_p/n} \text{sh}^2\ \eta\ d\eta \int_0^{h_p/n} v^2(\eta)\ d\eta\\ = \int_0^{h_p/n} \Big(\int_{s_p}^{\pi/2} \frac{\partial\psi(s+i\eta)}{\partial s}\ ds\Big)^2\ d\eta \quad (1)\end{aligned}$$

Now using the Cauchy-Buniakovski inequality, we have

$$\begin{aligned}\cos^2 s_p \int_0^{h_p/n} \text{sh}^2\ \eta\ d\eta\\ \le c_p \int_0^{h_p/n}\int_{s_p}^{\pi/2} \Big(\frac{\partial\psi(s+i\eta)}{\partial s}\Big)^2\ dsd\eta\\ \le c_p \int_0^{h_p/n}\int_{s_p}^{\pi/2} |(\lambda(\zeta) - \sin\zeta)'|^2\ dsd\eta\\ = c_p \iint_{D^*} |\lambda(\zeta) - \sin\zeta)'|^2\ dsd\eta\\ \le c_p \iint_{\Pi_+} |(\lambda(\zeta) - \sin(\zeta))'|^2\, dsd\eta\\ = c_p \iint_{K_+(h)} |z'(k) - 1|^2\, dudv = c_p\pi Q_0,\end{aligned}$$

where $D^* = [s_p, \pi/2] \times [0, h_p/n]$.

Thus

$$\int_0^{h_p/n} \text{sh}^2\ \eta\ d\eta \le CQ_0,$$

where $C = c_p\pi/\cos^2 s_p$. After integration we have

$$\frac{\text{sh } 2h_p/n}{4} - \frac{h_p}{2n} \le CQ_0.$$

For the function $\phi(h_p) = (\text{sh } 2h_p/n)/4 - h_p/(2n)$, there exist certain constants $C_1, C_2 > 0$, such that

$$\begin{aligned}C_1(h_p^{1/2} + h_p^{1/3}) \le \phi(h_p)\\ \le C_2(h_p^{1/2} + h_p^{1/3}), \quad h_p \in (0 + \infty)\end{aligned}$$

so

$$H_p \le C\Big((\pi c_p Q_0)^{1/2} + (\pi c_p Q_0)^{1/3}\Big),$$

where $C = C(C_1, C_2)$, $c_p = (n-p)\pi/n$. ■

4. The Jacobi Matrix

Let H be a Hilbert soace whose elements are vectors

$$y = (\ldots, y_{-k}, \ldots, y_1, y_0, y_1, \ldots, y_k, \ldots)$$

with the scalar product

$$(x, y) = \sum_{i=-\infty}^{+\infty} x_i y_i.$$

Let us consider a self adjoint bounded operator A in H, associated to the infinite periodic Jacobi matrix ($\mathcal{J}$-matrices)

$$A = \begin{pmatrix} . & . & . & . & . & . & . & . \\ b_{-1} & a_0 & b_1 & 0 & . & . & . & . \\ 0 & b_0 & a_1 & b_1 & 0 & . & . & . \\ . & 0 & b_1 & a_2 & b_2 & 0 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & 0 & b_{k-1} & a_k & b_k & 0 \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

The elements $a_k = a_{k+n}$ are positive, and the elements $b_k = b_{k+n}$ are negative.

The spectrum S of this operator has a finite number of zones:

$$\left[\mu_k^+, \mu_{k+1}^-\right], \ \ (k = 0, 1, ..., n-1) : S = \bigcup_{k=0}^{n-1} \left[\mu_k^+, \mu_{k+1}^-\right],$$

where

$$\mu_0^+ = \mu_0^- = \mu_0, \quad \mu_n^+ = \mu_n^- = \mu_n^-$$

and

$$\mu_0 < \mu_1^- \le \mu_1^+ < 0 \cdots < \mu_{n-1}^- \le \mu_{n-1}^+ < \mu_n.$$

The equatioin $Ay = \mu y$ is equivalent to the second order difference equation

$$b_{k-1}y_{k-1} + a_k y_k + b_k y_{k+1} = \mu y_k \quad (k = 0, \pm 1, \ldots).$$

Let us represent the fundamental system of solutions by

$$\begin{aligned}P(P_-(\mu), P_0(\mu), P_1(\mu), \ldots)\\ \text{and } \ Q(Q_{-1}(\mu), Q_0(\mu), Q_1(\mu), \ldots),\end{aligned}$$

subject to the initial conditions

$$\begin{cases} P_{-1}(\mu) = 0, & Q_{-1}(\mu) = 1 \\ P_0(\mu) = 1, & Q_0(\mu) = 0. \end{cases}$$

For these polynomials holds the analogous of the Liouville-Ostragradski formula

$$P_{k-1}(\mu)Q_k(\mu) - P_k(\mu)Q_{k-1}(\mu) = -\frac{b_{-1}}{b_{k-1}}, \quad (k = 0, 1, \ldots).$$

The numbers $\mu_k^{\pm}$, $(k = 0, \ldots, n)$ are the roots of the equation $u_+(\mu) = \pm 1$, where

$$u_+(\mu) = \frac{P_n(\mu) + Q_{n-1}(\mu)}{2}$$

and

$$S = \{\mu : u_+^2(\mu) \leq 1\}.$$

Proposition 6. *Let $P_n(z)$ be a polynomial of degree n. Then $P_n^2(z) = 1$ has only real roots if and only if $P_n(z) = \cos\theta(z)$, where $\theta(z)$ is a conformal transformation of the upper half plane onto the region $D_n(h)$, such that $\theta(\infty) = \infty$.*

Let $k(z)$ be the inverse function of $z(k)$ and considering that $z(k) = \theta^{-1}[n(\arcsin k + \pi/2)]$, then $k(z)$ can be written as

$$k(z) = \sin\left(\frac{\theta(z)}{n} - \frac{\pi}{2}\right), \; z \in \mathbb{C}_+.$$

That is $k(z) = -\cos\theta(z)/n$, $z \in \mathbb{C}_+$. Let us point out that $k(z)$ is analytic in $\mathbb{C}_+$, continuous in $\mathbb{C}_+$ and can be continued to $\mathbb{C}_-$ by the principio de simetría de Riemann-Schwarz symmetry principle by the formula $k(z) = \overline{k(\overline{z})}$.

Therefore, the following theorem holds.

Theorem 7. *Let $k(z)$ be a conformal transformation of $\mathbb{C}_+$ onto the region $K_+(h)$. Then it happens that*

$$k(z) = z + \frac{Q_0}{z} + o\left(\frac{1}{z}\right), \quad z \to \infty,$$

where

$$Q_0 = \frac{(n-1)}{n+1}\left[2(b_0 b_1 \ldots b_{n-1})\right]^{2/n} + \frac{1}{n}\left(\sum_{i=0}^{n-2}\sum_{j=1}^{(n-1)-i} a_i a_j - \sum_{i=0}^{n-1} b_i^2\right),$$

and a_i, b_i are entries of the Jacobi matrix.

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