PACS №: 03.30.+p;03.65.Pm;11.30.Cp;41.20.-q
Zbigniew Oziewicz[1,2]
Universidad Nacional Autónoma de México
Facultad de Estudios Superiores Cuautitlán
Apartado Postal # 25, C.P. 54714 Cuautitlán Izcalli Estado de México,
e-mail: oziewicz@servidor.unam.mx

Electric and Magnetic Fields, within Categorical Kinematics

Contents

Abstract

The categorical kinematics is a groupoid-*category* of the massive observers, with binary relative velocities as the invertible morphisms, instead of the isometric Lorentz transformations. In categorical kinematics the *inverse* relative velocity, $\mathbf{v}^{-1}$, must be interior-observer-dependent, and no more necessarily absolute as in the Einstein's isometric formulation, where $\mathbf{v}^{-1} \equiv -\mathbf{v}$. Within the categorical kinematics the transformation of the electric and magnetic fields relative to the moving observer, is slightly different, when compared with the transformations deduced by Heaviside in 1888-1889, derived by Lorentz in 1904, and by Einstein in 1905.

The adopted differential co-frame of not necessarily inertial observer, is not unique. The general transformations among adopted co-frames of observers are derived in terms of the binary relative velocity. These frame-transformations generalize the Robertson's test-theory of special relativity, for the categorical kinematics.

The main conclusion: observer-independence/dependence and the Lorentz-invariance/covariance are different concepts. The same statement holds in Newtonian physics: observer-independence is not be the same as the Galilean-group-invariance.

[1]Supported by el Consejo Nacional de Ciencia y Tecnología (CONACyT de México), Grant # U 41214 F.
[2]A member of Sistema Nacional de Investigadores in México, Expediente # 15337.

> Two contradictory theories (explaining the same reality), if not considered as foundations of foundations, can be both useful for the research.
>
> Henri Poincaré in Preface to 'Electricity and Optics', Paris 1890,1901.

1. Coordinate Transformations in XIX Century

There is wide spread believe that the Lorentz/Poincare group is a symmetry group of the Maxwell and of the d'Alembert wave equations for the massless radiation, see e.g. [Sommerfeld 1949, 1952 p. 223]. In fact, the Lorentz/Poincare group is an isometry group (symmetry group) of the metric-tensor for *empty* spacetime. Instead, the Maxwell and d'Alembert wave equations are invariant with respect to the 15-dimensional conformal group.

Another wide spread dogma is that the relativity transformations among *massive* reference-systems must be obligatory the symmetry of the *massless* Maxwell equations. If this would be the case indeed, then the conformal group would be the relativity group instead of the Poincaré group.

The special relativity is usually defined by a single postulate, that any two inertial massive reference systems are connected by inhomogeneous Lorentz transformation, see e.g. [Schrödinger 1931; Bargmann 1957, p. 161]. We are trying to oppose to this dogmatic trend, by arguing that the *massive* reference-systems need not be connected by an isometry, because, among other, the Poincaré symmetry group is the symmetry group of the metric tensor of the *empty* energy-less spacetime, [Oziewicz 2005, 2006, 2007].

Let's first state the conformal invariance of the massless radiation, that we will, later on, oppose to the concept of a massive observer and velocity-dependent non-isometric transformation among observers.

The Hodge star tensor field, $*_g$, is concomitant of the metric-tensor field, $g = g^*$, the Einstein gravitational potential, and depends on the choice of the orientation, see e.g. [Harnett 1991, 1992; Oziewicz 1994; Cruz and Oziewicz 2003]. The divergence differential operator acting on differential forms, $\delta_g \equiv *_g^{-1} \circ d \circ *_g$, does depends on the Hodge star tensor field $*_g$.

In what follows $g^\wedge$ is not natural isomorphism of Grassmann algebras, intertwining Hodge's stars,

$$g^\wedge \circ *_g = *_g \circ g^\wedge.$$

We define the first-order differential operators acting on contravariant multi-vector fields, the gradient, divergence and rotation, as follows,

$$\begin{aligned} \operatorname{grad}_g &\equiv g^{-1\wedge} \circ d \circ g^\wedge, \\ \operatorname{div} &\equiv g^{-1\wedge} \circ \delta_g \circ g^\wedge, \\ \operatorname{rot}_g &\equiv g^{-1\wedge} \circ *_g^{-1} \circ d \circ g^\wedge. \end{aligned} \tag{1}$$

Throughout this paper, F denotes the closed differential bi-form of the electromagnetic field (the electromagnetic strength), $dF = 0$, and $\mathbf{F}$ is the electromagnetic bi-vector field such that $F = g^\wedge \mathbf{F}$.

1. Theorem (Cunningham 1909/1910, Bateman 1910). *The Maxwell equations in four-dimensions,* $dF = 0$ *&* $\delta_g F = 0$, *equivalently,* $\operatorname{grad}_g \mathbf{F} = 0$, *and* $\operatorname{div} \mathbf{F} = 0$, *are conformal-invariant.*

Proof. For every positive scalar field, $f > 0$, the Hodge star on differential forms of a grade $k \in \mathbb{N}$, for conformal-scaled metric, $g \longmapsto fg$, in n-dimension, is as follows

$$*_g \quad \longmapsto \quad *_{fg} = f^{\frac{n}{2}-k} *_g . \tag{2}$$

Therefore for, $n = 2k$, the Hodge's star is conformal-invariant, $*_{fg} = *_g$, and this imply the above known theorem. ■

Weitzenböck in 1923 realized that the d'Alembert-Laplace-Beltrami-De Rham wave operator, is coordinate-free and basis-free. This differential operator does not needs necessarily Christoffels, and, on differential multi-forms, and on multi-vector fields (contra-variant), is defined appropriately as follows

$$\begin{aligned} \triangle_g &= (d + \delta_g)^2 = \delta_g \circ d + d \circ \delta_g, \\ g^{-1\wedge} \circ \triangle_g \circ g^\wedge &= (\operatorname{grad}_g + \operatorname{div})^2. \end{aligned} \tag{3}$$

2. Theorem (Voigt 1887). *In two-dimensions the d'Alembert-Laplace wave equation* $\triangle_g \phi = 0$ *for the scalar field* ϕ, *is conformally invariant.*

This conformal invariance was demonstrated by Voigt in 1887 assuming implicitly that the metric-tensor g is GL-covariant, and using the coordinate transformations. Voigt's proof is reproduced at the very end of the present Section.

Coordinate-free proof. For two-dimension, $n = 2$, $*_{fg} = f^{1-k} *_g$. The divergence differential operator on the Pfaffian differential one-forms is, $\delta_{fg} = (1/f)\delta_g$, and the wave operator on scalar fields (scalar fields are divergent-less) is

$$\triangle_{fg} \equiv \delta_{fg} \circ d = \frac{1}{f}\, \delta_g \circ d = \frac{1}{f}\, \triangle_g. \tag{4}$$

The wave-*operator* acquire the scalar factor, therefore the d'Alembert-Laplace wave-*equation* is conformally invariant. ■

Now in XXI century the majority of the physics community are not familiar with coordinate-free discussion. A tensor as a coordinate-free concept is still alien, and coordinate-free reasoning is considered as a defect that is 'enveloping the physical ideas' (cited from secret Referee Report). In fact, it is just contrary: coordinate-dependent reasoning envelops the physical ideas and confuses. The Physics is coordinate-free. Nowadays almost everybody demands the presentation explicitly in 'sacred' coordinates and bases-dependent matrices of XVIII and XIX centuries. Coordinates and matrices (matrix of the boost, Dirac matrix, etc) are considered as the only legitimate language of physics. Every coordinate transformation must be suspect if the same transformation can not be re-understood in coordinate-free way. The obscure meaning of the coordinate systems was criticized already by Hermann Grassmann in 1844 (mine translation):

> It is necessary to abandoner damn-all habit introducing arbitrary coordinates that have nothing to do with the matter subject and darken ideas: calculus become thoughtlessness mechanical set of strange formulas, and this is mortal for the spirit.
>
> Hermann Grassmann, 1844, Chapter 1.

For Grassmann, Heaviside, as well as for Gibbs, the vector is basis-free. The concept of a tensor introduced Voigt in crystallography.

The Galilean transformations of the space coordinates were considered by Newton. A mathematical coordinate-time must not be confused with the coordinate-free proper time introduced by Hermann Minkowski in 1908. The coordinate-time transformation were considered firstly by Potier in 1874 and by Voigt in 1887 [Voigt 1887, Ernest and Hsu 2001]. Nowadays, neither Potier, nor Voigt, are accepted as the forerunners of special relativity of time, notwithstanding that they introduced relative coordinate-time (local time), because they consider *non*-reciprocal coordinate time transformation,

$$\mathbf{v} \quad \not\longleftrightarrow \quad -\mathbf{v}.$$

In the categorical kinematics, the transformations among reference systems (among Zel'manov's monads [Zel'manov 1976]) do not satisfy reciprocity (similarly as in Potier's and in Voigt's transformations), contrary to the Einstein's special relativity with exactly reciprocal Lorentz isometric transformations, [Oziewicz 2005,2006,2007].

Albert Einstein started his 1905-paper with two postulates:

- Relativity principle: the physical equivalence of the rest with motion, must be attributed to Nicola Copernico and Giordano Bruno. Galileo in 1632: the physical laws, phenomena, are observer-independent. The absolute preferred-rest does not exists. Emphasized by Poincaré in 1904. Einstein added: the electromagnetic phenomena are also preferred-rest-free.
- The speed of the light is observer-independent, is emitter-free.

Schrödinger in 1931, and Bargmann in 1957, among many other, re-phrased the Einstein two postulates, as the following precise mathematical single postulate: inertial frames of reference *must* be connected by Lorentz transformation. We consider that this is the naive simplification.

Léon Brillouin expressed opinion that Lorentz-invariance is independent of relativity principle, 'Lorentz transformation is definitely not physical' [1970, page 71]. Gill and Lindesay also observed in 1993 that, a priori, two Einstein's postulates have nothing to do with the Lorentz-invariance. Gill and Lindesay call the Lorentz-invariance, the *third postulate* [Gill and Lindesay 1993]. Gill and Lindesay accuse Minkowski [1908, 1909] of the third postulate, however, the Lorentz transformations of coordinates were deduced by Einstein in 1905, who postulate the 'obvious' reciprocity principle for relative velocities, $\mathbf{v}^{-1} \equiv -\mathbf{v}$. We consider that this reciprocity axiom alone is the most important axiom of the Einstein special relativity theory. It is self-hidden axiom, being 'obvious'. The logical sequence of the Einstein deduction of the relativity transformations in 1905 can be re-expressed as follows.

The reciprocity principle is necessary for the isometric Lorentz boost transformation,

$$L(\mathbf{v}) \text{ is isometry} \quad \Longrightarrow \quad \{L(\mathbf{v})\}^{-1} = L(-\mathbf{v}) \qquad (5)$$

The associative composition of the Lorentz boosts imply non-associative composition of the relative velocities [Ungar 1988, 2001]. Then, the second Einstein's postulate, about the emitter-free speed of the light, is *Corollary* from addition of velocities.

Einstein in 1905 used explicitly light-cone invariance only. The metric-tensor was introduced by Minkowski in 1908 who observed that the Lorentz transformations of coordinates deduced by Einstein, are isometries of metric tensor.

Voigt in 1887 was primary motivated by explaining the Michelson & Morley experiments in 1881 and in 1887. Voigt was the first who proposed the universal speed of the light [Wesley 1986; Ernest & Hsu 2001].

The metric tensor fields, a covariant g, acting on vector fields, and a contravariant g^{-1}, acting on differential forms (Einstein's gravity potentials), are coordinate-free & basis-free. Following Riemann [1873], Minkowski [1908], Einstein [1913], Cattaneo [1958], Mitskievich [1996, 2006], and following many present textbooks, the metric tensor field is commonly denoted by $(g =)\ ds^2 = (ds) \otimes (ds)$, and referred to as 'the line element', or 'the invariant interval', or 'a quadratic form'. Such unfortunate notation could suggest incorrectly that this metric tensor is simple, *i.e.*, it is degenerate with no inverse, and this is not the case. For example, Mitskievich in his recent monograph [Mitskievich 2006], stress that ds is not total differential. The point is that the differential Pfaffian one-form, ds, does not exists at all (closed or not closed), because the tensor field g must be entangled, $g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \ldots$, with $\omega^1 \wedge \omega^2 \wedge \ldots \neq 0$. To be invertible, can not be simple tensor field in terms of only single differential one-form, $g \neq \omega \otimes \omega$.

The Voigt coordinate-proof in 1887. The Voigt coordinate proof of Theorem 2. is as follows. Let

$$g = -c^2 dt \otimes dt + dx \otimes dx.$$

This imply that, $*_g\, dx = dct$, and, $*_g\, dct = dx$. Then, the d'Alembert-Laplace operator restricted to scalar fields, has the following coordinate form

$$\triangle_g = \delta_g \circ d = -\left(\frac{\partial}{\partial ct}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2. \tag{6}$$

Voigt in 1887 implicitly consider that tensor g is GL-covariant relative to the GL-action on module of the differential one-forms. If $h \in GL$, then, $h^{\otimes}$, is isomorphism of the tensor algebra, and in general, $h^{\otimes} g \neq g$. Voigt consider the following transformation, $h \in GL2$, with real constants

$$\begin{gathered} v,\ a \neq 0,\ c \neq 0, \quad 0 \neq f \equiv a^2(1 - (v/c)^2) \in \mathbb{R}, \\ x' \equiv a(x - vt), \quad t' \equiv a(t - vx/c^2), \\ dt' \wedge dx' = f dt \wedge dx. \end{gathered} \tag{7}$$

Here, $dt' \wedge dt \neq 0$. Therefore Voigt's local coordinate times, t' and t, are relative. Hence the inverse transformation for $f \neq 1$ is not reciprocal,

$$\begin{aligned} x &= af^{-1}(x' + vt') \\ t &= af^{-1}(t' + vx'/c^2), \end{aligned} \tag{8}$$

$$\{x, t, x', t', v\} \quad \not\Longleftrightarrow \quad \{x', t', x, t, -v\}, \tag{9}$$

$$g \longmapsto h^{\otimes} g = -c^2 dt' \otimes dt' + dx' \otimes dx' = fg \tag{10}$$

$$\triangle_g = f\left\{-\left(\frac{\partial}{\partial ct'}\right)^2 + \left(\frac{\partial}{\partial x'}\right)^2\right\} = f \triangle_{h^{\otimes} g}.$$

The *operator* acquire scalar factor, and therefore the d'Alembert-Laplace wave-*equation* is invariant relative to the Voigt transformation (7), and this proves Voigt's Theorem 2. ■

Moral: the conformal invariance of the Laplace wave operator (in two-dimensions) do not need reciprocity.

2. Metric Field as a Morphism

Let $\mathcal{F}$ be an algebra of scalar fields, $x, t \in \mathcal{F}$. Then, the $\mathcal{F}$-module of the vector fields is the Lie $\mathcal{F}$-module of derivations, $\operatorname{der}\mathcal{F}$. In what it follows, $g = g^*$, denotes a metric-tensor field of the Lorentzian signature $\{-, +++\}$. We do not need to suppose that g must be necessarily curvature-free. A metric-tensor is interpreted as the scalar-field-valued 'product' on 'pairs' of vector-fields. Usual practice is to denote a square of a magnitude of a vector field $X \in \operatorname{der}\mathcal{F}$, or a square of a magnitude of a differential one-form α, (accordingly, a magnitude of a vector and a covector), by a scalar field (accordingly a real number), X^2 and $\alpha^2 \in \mathcal{F}$. This makes incorrect impression that a magnitude is a property of a vector or covector only. In fact, a square of a magnitude of a vector field, $X^2 \equiv g(X \otimes X)$, is a function of *two* independent tensorial variables. Depends on a vector field X, and depends on a choice of a metric tensor g (a choice of a scalar product). The most elementary example, the different choices of the units without changing a vector, the same car velocity in miles/hour and kilometres/hour, is nothing but the two different choices of the scalar product.

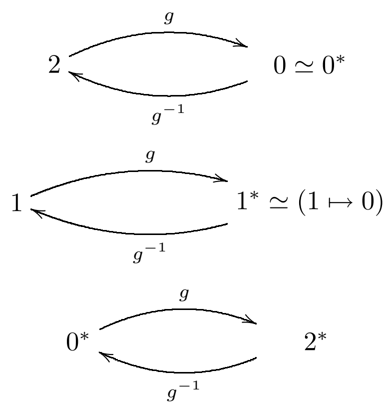

Fig. 1. Scalar product as a morphism

Both tensor fields, g and $g^{-1} \in (\operatorname{der}\mathcal{F})^{\otimes 2}$, lift to the isomorphisms of tensor and Grassmann algebras. There are still several interpretations of these metric tensor fields, see Figure 1. We denote by $0 \simeq 0^* \in \mathbb{N}$ the grade (rank, arity) of the scalar fields, the grade of the module of the Pfaffian differential one-forms is 1^*, the tensor product of Pfaffian differential one-forms has the grade 2^*. Correspondingly, the dual module of the one-vector fields has assigned the grade$(\operatorname{der}\mathcal{F}) = 1$, The second rank contra-variant tensors have the grade 2. For this non-commutative free monoid on two generators, 1 and 1^*, we need two copies of natural numbers, $\mathbb{N}$ and $\mathbb{N}^*$. A scalar product g, is defined to be non-degenerate morphism of an arity, $2 \mapsto 0$. With this notation there are the following type interpretations of the metric tensor fields as shown on Figure 1. We will use these several interpretations of metric tensor fields as the morphisms. A scalar product considered as a morphism from vectors to covectors, $g \in (1 \mapsto 1^*)$, is genuine *invertible* isomorphism (not natural).

When talking about metric tensor field g, most often only the first interpretation is spelled, $2 \mapsto 0$. One can pass from one to another interpretation, Figure 1, in terms of evaluation and coevaluation.

The above coordinate-free algebraic interpretations of the metric tensor field, we oppose to the Helmholtz's subjective relativity of the metric, where an observer is identified with a coordinate system, an observer = coordinate inhabitant, see [Honig 1977].

3. Observer

In order to have just one space, and one proper-time, we need to fix one massive body as the reference system. The correspondence, massive body $\leftrightarrow$ idempotent, is motivated by a desire that every massive body, an endomorphism field, $p \in \operatorname{End}(\operatorname{der}\mathcal{F}) \simeq (1 \mapsto 1)$, splits

$$\operatorname{der}\mathcal{F} = (\ker p) \oplus (\operatorname{im} p) = (\text{space}) \oplus (\text{time}). \tag{11}$$

The choice of one body, for example the Earth, as the reference system, does *not* need coordinates. Such choice is coordinate-free, and basis-free. We call any massive body, the Earth, the Moon, an observer. However no measuring devices, rods and clocks are involved.

Therefore each massive observer is identified with trace-class (transversal) idempotent, an endomorphism, $(1, 1^*)$-tensor field, $p^2 = p \in \operatorname{End}(\operatorname{der}\mathcal{F})$, with $\operatorname{tr} p = 1$,

$$p^2 = p \quad \Longrightarrow \quad \operatorname{tr} p = \dim \operatorname{im} p. \tag{12}$$

A pull back (transpose) of p, is denoted by p^*. It is a $(1^*, 1)$-tensor field. The composition, $p \circ g^{-1}$, and their transpose, $(p \circ g^{-1})^* = g^{-1} \circ p^* \in (1^* \mapsto 1)$, are both morphism with the same domain and codomain, and therefore one can ask that an idempotent p (a massive body), is metric-compatible (g-orthogonal). We postulate the following conditions that are equivalent that tensor p is simple. Then, an idempotent p, can be expressed in terms of the non-zero p-eigenvector field $P \in \operatorname{der}\mathcal{F}$.

$$p^2 = p \quad \& \quad \operatorname{tr} p = 1 \quad \& \quad g^{-1} \circ p^* = p \circ g^{-1} \quad \Longleftrightarrow \quad p = \frac{P \otimes (gP)}{g(P \otimes P)}. \tag{13}$$

Metric-compatible observer is conformally invariant. An idempotent is said to be time-like if his every non-zero eigenvector is time-like (in Zelmanov's terminology: a monad field [Zel'manov 1976, Mitskievich 2006]), $pP = P \neq 0 \;\Rightarrow\; P^2 < 0$. The space-like simultaneity of an observer p, must be given by the Einstein-Minkowski proper-time, identified here with Pfaffian differential form gP/P^2. This time-like differential Pfaff form, gP/P^2, encode the unique empirical and metric-dependent simultaneity of an observer p.

The notion of a basis-free single observer as the $(1, 1)$-endomorphism field, as used here, was introduced independently by many Authors, [Świerk 1988, Fecko 1977; Kocik 1997; Gottlieb 1997]. For an algebra generated by many observers we refer to [Cruz and Oziewicz 2003, 2006; Oziewicz 2007].

4. Binary Relative Velocity

A set of all relative velocities among massive bodies is not a vector space of linear algebra. Neither every pair of such relative velocities can be composed (velocity addition is the partial operation), nor the commutativity of such composition, if the composition is possible, makes sense. This holds equally well in Newtonian physics, and as well in relativistic physics with the finite velocity of light.

Consider two-body massive system,

$$p \equiv \frac{P \otimes (gP)}{P^2}, \quad \text{and} \quad q \equiv \frac{Q \otimes (gQ)}{Q^2}, \qquad \operatorname{tr}(pq) = \frac{(P \cdot Q)^2}{P^2 Q^2}. \tag{14}$$

In what follows, we set all monad fields, the non-zero time-like eigenvectors of observers, to be normalized, $pP = P \in \operatorname{der}\mathcal{F}$, with $P^2 = -1$. Then, $1 \leq (P \cdot Q)^2$.

3. Definition (Binary relative velocity). Let c be a limited scalar speed. A velocity-morphism from an observer p to observed q, is defined as follows

$$\frac{\varpi_g(P,Q)}{c} \equiv \frac{(\operatorname{id} - p)Q}{(-gP)Q} = \frac{Q}{-P \cdot Q} - P, \tag{15}$$

$$\left(\frac{\varpi_g(P,Q)}{c}\right)^2 = 1 - \frac{1}{(P \cdot Q)^2}. \tag{16}$$

One can see, that, the above velocity-morphism is a space-like vector field, interpreted as a velocity field of q relative to p.

The relative velocity is defined above as the isomorphism in the category of the massive observers. In Figure 2, we use Definition 7., for the traceless arrows. In this Figure, the relative velocities-isomorphisms-arrows, $\varpi(p, r)$ and $\varpi(p, s)$, are not composable arrows, and therefore can not be added. The kinematics in terms of the relative velocities-isomorphisms, is said to be categorical. The categorical kinematics must be contrasted with the isometric Lorentz transformation, where the addition of relative velocities is global within the Lorentz group.

Definition 3. of the relative binary velocity, formula (15), is implicit in [Minkowski 1908] (who call this formula 'the Lorentz transformation'). In adopted system of coordinates this formula appears in [Vargas 1982, p. 770, formula (29)]. Definition 3. of space-like relative binary velocity, formula (15), as a coordinate-free and basis-free projection field, we proposed in [Świerk 1988], where we observed that such relative velocity field can *not* parameterize the isometric Lorentz boost. The same definition was introduced independently by many Authors, Matolcsi [1993, p. 191], Bini, Carini and Jantzen [1995], Gottlieb [1997], Matolcsi and Goher [2001, p. 89, Definition (18)], Mitskievich [2005, formula (4.19) on p. 16], Bolós [2005]. See also [Oziewicz 2005, 2006, 2007] and [Page 2006]. We are using tensor fields, instead, Synge in 1960, and Bolós in 2005–2006, are using parallel transport in order to define relative velocity.

Let space-like velocity, $\mathbf{v}/c \equiv \varpi_g(P,Q)$ be the binary velocity of the body q relative to an interior observer p. This means that $\mathbf{v} \in \ker p$, $p\mathbf{v} = 0$. Then we display this velocity $\mathbf{v}$ as a categorical arrow (morphism, directed-path) which originates (is outgoing) at observer p (p is a node of the directed graph), and terminates (is incoming) at an observed body q,

$$\cdots \longrightarrow \quad p \;\underset{\mathbf{v}^{-1}}{\overset{\mathbf{v}}{\rightleftarrows}}\; q \quad \longrightarrow \cdots; \qquad (17)$$

Observer of $\mathbf{v}$ is p, observed body with $\mathbf{v}$ is q, observed body with $\mathbf{v}^{-1}$ is p. Observer of $\mathbf{v}^{-1}$ is q, $\mathbf{v}^{-1} \in \ker q$.

The binary relative velocity field needs *not* to be constant, and Definition 3. holds for the most general observers (not necessarily inertial). Within the categorical kinematics there is no need for distinction among 'special' and 'general' relativistic kinematics.

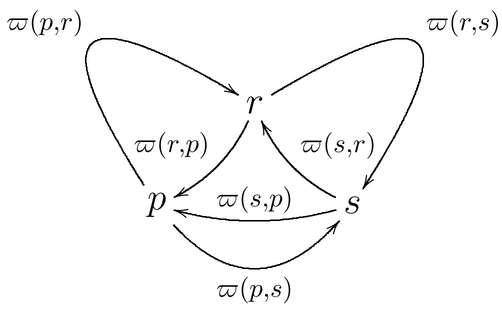

Fig. 2. A system of three massive bodies, p, r, s, with six binary velocities-isomorphisms.

4. Corollary (Light velocity). The light propagation is governed by Maxwell equations and is given by the light-like vector field, $L^2 = 0$. When the limiting velocity is identified with the speed of the light, then, we deduce that the magnitude of the speed of the light is observer-independent,

$$\mathbf{c} \equiv \varpi_g(P,L) = \left(\frac{L}{-P\cdot L} - P\right)c \qquad \Longrightarrow \quad \mathbf{c}^2 = c^2. \qquad (18)$$

Textbooks teach that 'all *inertial* observers measure the same speed of the light', cf. with [Matolcsi 1993, §4.7.2; Dvoeglazov & Quintanar González 2006, Conclusions]. Such statement could suggest that non-inertial observers would measure observer-dependent speed of light. Corollary 4. is stronger: categorical kinematics predict that speed of the light is independent of *arbitrary* observer, including *all* non-inertial, rotating and accelerating observers. Like in Einstein's gravity theory.

5. Corollary (Heaviside-FitzGerald-Lorentz factor). Within the categorical kinematics the following formulas hold. Set, $\mathbf{v}/c \equiv \varpi_g(P,Q)$, and, $\mathbf{v}^{-1}/c \equiv \varpi_g(Q,P)$. Hence

$$P\cdot\mathbf{v} = 0, \quad Q\cdot\mathbf{v}^{-1} = 0, \qquad (19)$$

$$\gamma_{\mathbf{v}} \equiv \frac{1}{\sqrt{1-\mathbf{v}^2/\mathbf{c}^2}} = -P\cdot Q, \qquad (20)$$

$$Q = \gamma_{\mathbf{v}}\left(\frac{\mathbf{v}}{c} + P\right), \quad P = \gamma_{\mathbf{v}}\left(\frac{\mathbf{v}^{-1}}{c} + Q\right), \qquad (21)$$

$$\frac{\mathbf{v}}{c} = \frac{Q}{\gamma_{\mathbf{v}}} - P = -\gamma_{\mathbf{v}}\frac{\mathbf{v}^{-1}}{c} - \left(\gamma_{\mathbf{v}} - \frac{1}{\gamma_{\mathbf{v}}}\right)Q = -\frac{\mathbf{v}^{-1}}{\gamma_{\mathbf{v}}c} - \left(1 - \frac{1}{\gamma_{\mathbf{v}}^2}\right)P, \qquad (22)$$

$$\frac{\mathbf{v}^{-1}}{c} = \frac{P}{\gamma_{\mathbf{v}}} - Q = -\gamma_{\mathbf{v}}\frac{\mathbf{v}}{c} - \left(\gamma_{\mathbf{v}} - \frac{1}{\gamma_{\mathbf{v}}}\right)P = -\frac{\mathbf{v}}{\gamma_{\mathbf{v}}c} - \left(1 - \frac{1}{\gamma_{\mathbf{v}}^2}\right)Q. \qquad (23)$$

6. Theorem (Binary velocity is not reciprocal). *The binary velocity* (15) *can not be reciprocal because simultaneity is not absolute,*

$$|\mathbf{v}| = |\mathbf{v}^{-1}|, \qquad \left(\mathbf{v}^{-1}+\mathbf{v}\right)^2 = -2\frac{\gamma_{\mathbf{v}}^2-1}{\gamma_{\mathbf{v}}}c^2, \qquad \left(\mathbf{v}^{-1}\wedge\mathbf{v}\right)^2 = -\frac{(\gamma_{\mathbf{v}}^2-1)^3}{\gamma_{\mathbf{v}}^4}c^4. \qquad (24)$$

It is very important to note, that within categorical kinematics, for the space-time dimension higher that 2, and for the given space-like $\mathbf{v}$, a space-like inverse, $\mathbf{v}^{-1}$, see graph (17), is not unique! One can ask: how many there are inverses? This non-uniqueness is due to that the condition (19) for the given $\mathbf{v}$, does not determine the unique time-like vector field P. Therefore, each time-like vector field P, satisfying the condition (19), and normalized such that $P^2 = -1$, gives his *own* inverse, $\mathbf{v}^{-1} = \mathbf{v}^{-1}(\mathbf{v}, P)$, as given explicitly by formula (23),

$$\mathbf{v}^{-1}(\mathbf{v}, P) = -\gamma_{\mathbf{v}}\mathbf{v} - \left(\gamma_{\mathbf{v}} - \frac{1}{\gamma_{\mathbf{v}}}\right) cP. \qquad (25)$$

This must be contrasted with the kinematics of the special relativity, where the inverse is unique and absolute, $\mathbf{v}^{-1} = -\mathbf{v}$.

7. Definition. To shorten notation in formula (26) below, we write ϖ instead of nilpotent endomorphism

$$\varpi(p,q) \equiv \varpi_g(P,Q) \otimes (-gP) \in (1 \mapsto 1),$$
$$\mathrm{tr}\ \varpi(p,q) = 0, \quad \{\varpi(p,q)\}^2 \equiv \varpi(p,q)\circ\varpi(p,q) = 0.$$

8. Corollary (Categorical boost). Let $c < \infty$. The coordinate-free and basis-free transformation among time-like bodies-idempotents in relative motion, the categorical boost, is as follows,

$$p \underset{\varpi(q,p)}{\overset{\varpi(p,q)}{\rightleftarrows}} q = q(p,\mathbf{v}) =$$
$$= \gamma_{\mathbf{v}}^2 \left\{ p + \frac{\varpi + g^{-1}\circ\varpi^*\circ g}{c} + \frac{\varpi\circ g^{-1}\circ\varpi^*\circ g}{c^2} \right\}$$
$$= \gamma_{\mathbf{v}}^2 \left\{ p - \frac{P\otimes g\mathbf{v} + \mathbf{v}\otimes gP}{c} - \frac{\mathbf{v}\otimes g\mathbf{v}}{c^2} \right\}, \qquad (26)$$

$$\begin{aligned} q^*\circ p^* &= -\gamma_{\mathbf{v}}\,(gQ)\otimes P, \\ q^*\circ(\mathrm{id}^* - p^*) &= -\gamma_{\mathbf{v}}\,(gQ)\otimes \mathbf{v}. \end{aligned} \qquad (27)$$

Proof. Insert formula (21) into (14). ■

The categorical kinematics is a groupoid category of the massive bodies, not necessarily inertial, with binary (interior) relative space-like velocities as the isomorphisms, instead of the isometric Lorentz transformations.

In categorical kinematics the reference systems are *not* related by isometric Lorentz transformations [Oziewicz 2005,2006]. The categorical boost, $p \mapsto q = q(p,\mathbf{v})$, formula (26), can *not* be induced from endomorphisms of the $\mathcal{F}$-module $\mathrm{der}\,\mathcal{F}$, because of the most essential constraints (19). It is exactly the constraints (19), that do not allows that exists endomorphism $K = K(\mathbf{v}) \in \mathrm{End}\,(\mathrm{der}\,\mathcal{F})$ (not necessarily isometry), such that, $K^{\otimes}p = q(p,\mathbf{v})$, where $q(p,\mathbf{v})$ as given by the categorical boost (26). If such hypothetical endomorphism K would exists, then we would be able to ask about K-covariance of all tensor algebra. However, the categorical kinematics with constraint (19), do not allows for the existence of such endomorphism at all.

We are going to apply above expressions, (26),(27), for coordinate-free derivation of the transformations of the electric and magnetic fields within categorical kinematics. We will show that categorical kinematics gives slightly different transformation of the electric and magnetic fields (different velocity-dependence), in comparison with the special relativity.

5. Lorentz Group versus Groupoid-Category

Some Authors use the name ‘groupoid’ as a synonym of a binary operation. This is not the case in the present paper, where groupoid is the name of a particular kind of *categories*. A category is said to be groupoid-category if every arrow is an isomorphism. A groupoid-category with exactly *one* object is said to be a *group*.

Reader could ask, how groupoid-category of massive observers, this category is denoted here by a symbol ϖ, with all morphisms given be categorical boost (26), is related with the concept of a group? How it is related with the Lorentz and Poincaré groups of symmetries (isometries) of the flat metric of the empty spacetime, of the Einstein’s special relativity [Einstein 1905]? In every group, and in every groupoid category, there is *always* the unique inverse operation. However, there are two differences among the concept of a group, and a concept of a groupoid category.

- The group binary associative operation is *global*. In the group every pair of elements can be composed. The morphisms in a category, if they can be composed, are composed associatively. However, in a category every morphism has the definite source object and the definite target object. Every physical relative velocity is a velocity of the concrete target body relative to another source body. Not all velocities-morphism are composable. The composition of categorical morphisms is the *partial* binary associative operation. For example, in categorical kinematics, we are *not* allowed to add the velocity of the Sun relative to Earth, with the velocity of the Moon relative to the Earth. These velocities-morphisms are not composable arrows. However such addition is allowed in Einstein’ special relativity where one can compose within the Lorentz group every Lorentz boost with another arbitrary Lorentz boost.

- Group possess the unique neutral element, e.g. the universal zero velocity. Contrary to this, in category of massive bodies, every massive body (every reference system) posses his *own* zero velocity-isomorphism.

In order to pass from categorical kinematics of the groupoid-category of massive bodies, to Einstein’s special relativity theory, it is necessary (but not sufficient) to *choose one massive* reference system ($e =$

Earth?) to be preferred. The choice of a preferred massive body will pick up the unique zero velocity, $\mathbf{0}_e$ (neutral element of a group). In this case all binary velocities, $\{\varpi(p,q)\}$, can be 'projected on' (measured by) the preferred body, and this leads to several distinct concepts of ternary relative velocities, $\varpi(e,p,q) \in \ker e$.

9. Definition (Ternary relative velocity). Let q, p, e, be a system of three massive bodies. The velocity of a body q relative to p, as measured/seen by an external (preferred) body e, $\varpi(e,p,q) \in \ker e$, is said to be the ternary relative velocity. Ternary velocity is said to be *reciprocal* if

$$\varpi(e,p,q) = -\varpi(e,q,p). \tag{28}$$

Ternary velocity is said to be isometric-image if

$$|\varpi(e,p,q)| = |\varpi(q,p)|. \tag{29}$$

In contrast to the unique binary relative velocity that is always the traceless arrow of the groupoid-category ϖ, and is never reciprocal, see Theorem 6. and formulas (24), there are *several* distinct concepts of the ternary velocities, depending whether this ternary-velocity as an arrow of ϖ, is, or it is not, reciprocal, traceless, isometric-image.

10. Theorem (Lorentz-boost-link problem). *The* reciprocal *ternary relative velocity is equivalent to the isometric Lorentz boost. Einstein's isometric exterior formulation needs ternary velocities that in general are not isometric-image, and possess non-associative addition.*

Proof. The proof was announced on several meetings [Oziewicz 2005, 2006]. The clue for understanding are: the reciprocity of the inverse velocity (*assumed* by Einstein in 1905), and the Lorentz-boost-link problem raised by van Wyk in 1986. ■

11. Clarification (The shortest history of the concept of velocity). Since Galileo, the velocity was defined in terms of the *measurement*, as the ratio of the distance to time, $\mathbf{v} = d\mathbf{x}/dt$. Such coordinate definition need the primary concepts: a space, a time, and the concept of a curve in the space [Lévy-Leblond 1980]. The most essential property of the physical velocity, is its *relativity*. This relativity is lost, or at least it is not explicit when using coordinates. One can ask: relative to what variable massive body ($\equiv$ time-like vector field), is this coordinate velocity $d\mathbf{x}/dt$? How $d\mathbf{x}/dt$ depends on reference system? Coordinates does not need massive bodies, and massive bodies does not need coordinates. The velocity of a moving car relative to a street, $\varpi(\text{street}, \text{car})$, is not the same as the velocity of the same car relative to itself, $\varpi(\text{car}, \text{car}) = \mathbf{0}_{\text{car}}$.

According to Einstein [1905] and Minkowski [1908, 1909], the velocity of the car relative to the street is given by Lorentz isometric transformation,

$$L(\mathbf{v})(\text{street}) = (\text{car}). \tag{30}$$

Van Wyk in 1986 was trying to solve the Lorentz-boost-link equation (30) for unknown velocity $\mathbf{v}$, for the given 'car' and the given 'street', and found that the solution is not unique! We showed elsewhere, that in order to fix one solution for the Lorentz-boost-link equation for $\mathbf{v}$ (30), one needs *third* massive body, an exterior observer [Oziewicz 2006]. That exterior massive observer is needed, it is clear also because the Lorentz transformation imply that the velocity is reciprocal, *i.e.* the velocity of the street relative to car must be, $-\mathbf{v}$, $L(-\mathbf{v}) \circ L(\mathbf{v}) = \text{id}$. Therefore the Einstein relative velocity among two reference systems, is ternary, and depends essentially on the arbitrary choice of the exterior observer, that was apparently not the intention of Einstein in 1905,

$$\mathbf{v} = \mathbf{v}(\text{exterior-observer}, \text{street}, \text{car}).$$

Within categorical kinematics, the Lorentz invariance/covariance is neither broken, nor violated. It is inapplicable, because the binary velocity-projection, Definition (15), can not parameterize the Lorentz isometric boost. Within categorical kinematics, the Lorentz/Poincaré/Galilei group-covariance/invariance is conceptually *separated* from observer-independence/dependence.

- The main problem of Einstein's isometric relativity: does the considered physical concept (e.g. spin-density, electromagnetic field, density of the electric charge, etc), is, or it is not, Lorentz-covariant, or Lorentz-invariant? What means Lorentz violation in field theory [Mattingly 2005]?
- The main problem of the categorical kinematics: does the considered physical concept, is, or it is not, observer-free?

6. Heaviside in 1888

Oliver Heaviside in 1888 presented the nine scalar Maxwell differential equations as the set of four, the ones we presently call Maxwell's equations, however they are not in Maxwell's *Treatise*.

The scalar factor, $\gamma_{\mathbf{v}}$ (20), was introduced by Oliver Heaviside in 1888, who observed how the electric and magnetic fields are related in two different reference systems in relative motion. The same factor, $\gamma_{\mathbf{v}}$, was introduced by George Francis FitzGerald in Dublin, in a letter to Heaviside in 1889, and independently by H. Antoon Lorentz in 1893, as the explanation of the nonexistence of the aether drift by Michelson and Morley in 1887. Lorentz denoted this factor by $k \equiv k(\mathbf{v})$, $1 \leq k$, $k^2(1-\mathbf{v}^2) \equiv 1$, and the Lorentz notation was adopted by Henrí Poincaré in 1905. Joseph Larmor in 1900 used his own denotation as $\sqrt{\varepsilon}$. The same factor appears in the Lorentz derivation in 1904 of the group of symmetry of the Maxwell equations. Albert Einstein in 1905 denoted this factor by $\beta \equiv \beta(\mathbf{v})$, in his derivation of the group transformation among coordinate frames (metric implicit). Starting 1970's most Authors established $\mathbf{v}^2/c^2 \longmapsto \gamma_\mathbf{v} \in [1,\infty) \subset \mathbb{R}$, [Mocanu 1986,1992],

$$\begin{aligned}&\gamma_\mathbf{v}^2\left(\mathbf{c}^2-\mathbf{v}^2\right)\equiv\mathbf{c}^2 \quad\text{and}\quad 0\leq\sqrt{\gamma^2-1}<\gamma,\\ &\gamma_\mathbf{0}\equiv 1\leq\gamma_\mathbf{u}<\gamma_\mathbf{c}=\infty.\end{aligned} \tag{31}$$

Let, $\mathbf{E}(p)$, be an electric field, and $\mathbf{B}(p)$, be orientation-dependent magnetic field, both these vector fields as measured by an observer p. Heaviside in 1888, deduced that these fields as measured by two different observers with velocity $\mathbf{v}$ of q relative to p, are related as follows,

$$\begin{aligned}\mathbf{E}(q)&=\gamma_\mathbf{v}\left\{\mathbf{E}(p)+\frac{\mathbf{v}}{c}\times\mathbf{B}(p)\right\},\\ \mathbf{B}(q)&=\gamma_\mathbf{v}\left\{\mathbf{B}(p)-\frac{\mathbf{v}}{c}\times\mathbf{E}(p)\right\}.\end{aligned} \tag{32}$$

The above Heaviside's formulas are re-derived by Hajra and Ghosh [2005, formulas (32) on page 66].

The Heaviside formula (32) formally coincide with the Lorentz boost transformation, [and actually with prediction of categorical kinematics, see below,] *only* in the case that the relative velocity $\mathbf{v}$ is perpendicular to the electric and magnetic fields, $\mathbf{v}\cdot\mathbf{E}(p)=0$ and $\mathbf{v}\cdot\mathbf{B}(p)=0$. We stress that there seems to be only formal coincidence of (32) with the isometric Lorentz boost as given by Einstein in 1905, see the next subSection below, because seems that Heaviside and Thomson consider *two* observers in mutual motion and does not used the Lorentz-covariance, and moreover, the actual coordinate transformations used by Heaviside and by Thomson, need not to be necessarily isometries. For more details we refer to [Heaviside 1888,1889,1893; Hajra & Ghosh 2005].

6.1. Lorentz in 1904, and Einstein in 1905

By definition, the Lorentz transformation is an isomorphism, $L \in \operatorname{End}(\operatorname{der}\mathcal{F})$, such that, the isometry condition holds, $L^\otimes g = g$. Here, $L^\otimes$, is an isomorphism of tensor algebra. Reader is invited to associate this isometry with Schrödinger's and Bargmann's formulation of the Einstein's special relativity [Schrödinger 1931; Bargmann 1957]. Special relativity is based on the axiom that the differential closed bi-form F of the electromagnetic field is Lorentz-covariant,

$$F \mapsto F(\mathbf{v}) \equiv \{L(\mathbf{v})\}^\wedge F.$$

The electric vector field, $\mathbf{E}$, and the magnetic pseudo-vector field, $\mathbf{B}$, are concomitant of two primitive tensor fields, are F-dependent and observer-dependent, $\mathbf{E}(F,p)$ and $\mathbf{B}(F,p)$. The best reference is [Kocik 1997], see also [Fecko 1997; Cruz & Oziewicz 2003]. The precise Definition 15., below, and expressions (46), following [Kocik 1997], are given again in the next Section. This understanding is opposed to almost all textbooks, where the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields, are considered, instead, to be the primitive independent 'variables', according to the historical pre-relativistic and non-logical opinion, and the electromagnetic field F is treated artificially to be $\mathbf{E}$- and $\mathbf{B}$-dependent, viz. $F(\mathbf{E},\mathbf{B})$. Thus, a photon F, is composed from (a mixture of) primitives $\mathbf{E}$- and $\mathbf{B}$-fields. See for example [Ungar 1991, formula (21) on page 580; Hehl & Obukhov 2003, page 123]. This is also reflected in the name of PACS number 41.20.-q. In fact, it is just the electromagnetic field F, and an material observer tensor field p, that are genuine independent, and an electric field $\mathbf{E}$ must be considered to be F-dependent and p-dependent, *i.e.*, $\mathbf{E}(F,p)$.

There is the logical inconsistency within the special relativity, observed by Ivezić in his publications since 1999. It is non-logical to consider that a biform F is Lorentz-covariant, and at the same time, treat the observer's time-like vector field (or the tensor field p), to be absolute Lorentz invariant.

12. Theorem (Lorentz boost). *Lorentz-covariant electromagnetic field F, and the fixed absolute observer field p, leads to the following textbooks Lorentz transformations of the electric and magnetic fields,*

$$F \xrightarrow{\text{Lorentz-boost}} F' \equiv F(\mathbf{v}), \tag{33}$$

$$\mathbf{E}(F,p),\mathbf{B}(F,p) \xrightarrow{\text{Lorentz-boost}} \mathbf{E}(F',p),\mathbf{B}(F',p), \tag{34}$$

$$\mathbf{E}(F')=\gamma_\mathbf{v}\left\{\mathbf{E}(F)+\frac{\mathbf{v}}{c}\times\mathbf{B}(F)\right\} - \frac{\gamma_\mathbf{v}^2}{\gamma_\mathbf{v}+1}\left\{\frac{\mathbf{v}}{c}\cdot\mathbf{E}(F)\right\}\frac{\mathbf{v}}{c}, \tag{35}$$

$$\mathbf{B}(F')=\gamma_\mathbf{v}\left\{\mathbf{B}(F)-\frac{\mathbf{v}}{c}\times\mathbf{E}(F)\right\} - \frac{\gamma_\mathbf{v}^2}{\gamma_\mathbf{v}+1}\left\{\frac{\mathbf{v}}{c}\cdot\mathbf{B}(F)\right\}\frac{\mathbf{v}}{c}. \tag{36}$$

Lorentz transformations (35),(36) imply among other,

$$\mathbf{v}\cdot\mathbf{E}(F')=\mathbf{v}\cdot\mathbf{E}(F), \quad \mathbf{v}\cdot\mathbf{B}(F')=\mathbf{v}\cdot\mathbf{B}(F). \tag{37}$$

We refer to the following original papers and textbooks where one can find the details of the derivation of the above transformations: [Lorentz 1904; Einstein 1905; Möller 1960; Panofsky and Phillips 1962; Jackson 1962, 1975, formula (11.149); Hestenes 1966; Landau & Lifshitz 1973; Jancewicz 1989,2000; Ungar 1991, pages 580–581; Dvoeglazov and Quintanar González 2006].

Ivezić [since 1999] observed correctly that all tensor fields are Lorentz-covariant, all tensor algebra must be Lorentz-covariant. Therefore an observer' time-like vector field P, is also Lorentz-covariant, and the tensor fields, F, p, and P, all must be Lorentz-covariant. The Ivezić version of special relativity is coined 'invariant special relativity = ISR', where, instead of (34)–(36), we have

$$\mathbf{E}(F,p),\mathbf{B}(F,p) \xrightarrow{\text{Lorentz-boost}} \mathbf{E}(F',p'),\mathbf{B}(F',p'). \quad (38)$$

Kotel'nikov, within the Galilean electrodynamics, propose still another moving-body transformation of the electric and magnetic fields [Kotel'nikov 1998]. Kotelnikov's transformation is different form the Lorentz transformation (35),(36), is different from Ivezić's proposal, and is different from what implies the categorical kinematics, as we consider in the next Section.

6.2. Isometry from Bivector

This Section is technical. We provide here the hint for a basis-free proof of the Lorentz transformation of the electric and magnetic fields, (35),(36).

Let α and β be differential multiforms. Then, we use the following notation for their Grassmann exterior product. The regular right representation of the Grassmann algebra in endomorphism algebra is denoted as follows,

$$\text{Grass} \xrightarrow{e} \text{End}\,(\text{Grass}), \quad e_\alpha\beta \equiv \alpha\wedge\beta. \quad (39)$$

Let, $P \in \operatorname{der}\mathcal{F}$, be a vector field, then, $i_P = (e_P)^* \in \operatorname{der}((\operatorname{der}\mathcal{F})^{*\wedge})$, is a graded derivation of the Grassmann algebra of differential forms. Let, $b \in (\operatorname{der}\mathcal{F}) \wedge (\operatorname{der}\mathcal{F})$, be a simple bivector field. Then, i_b, is the composition of two derivations, that is not derivation. However, i_b is a well defined contraction acting on differential multiforms. We will use the following assumptions and notations for a bivector b, and for a vector P,

$$b\wedge b = 0, \quad b^2 \leq 1, \quad \gamma_b \equiv \sqrt{1-b^2}, \quad (40)$$

$$V \equiv \frac{1}{1+\gamma_b} i_{gb}(P\wedge b) - i_{gP}b - \frac{b^2}{1+\gamma_b}P. \quad (41)$$

The Lie algebra of isometry group O_g is given by bivectors. Consider the isometry, $L_b \in O_g$, generated by a simple bivector b as assumed in (40). Then, $(L_b)^{-1} = L_{-b}$, and we can state the technical Lemma.

13. Lemma. *Using abbreviation* (41) *for a vector field V, the following commutator expression holds on multiforms,*

$$i_P \circ L_b^* - L_b^* \circ i_P = i_V + \frac{2}{1+\gamma_b} e(i_P gb) \circ i_b. \quad (42)$$

Consider the following data,

$$P^2 = -1, \quad P\cdot\mathbf{v} = 0, \qquad \gamma_{\mathbf{v}} \equiv \frac{1}{\sqrt{1-\mathbf{v}^2/\mathbf{c}^2}} \neq -P\cdot Q, \quad (43)$$

$$b = \gamma_{\mathbf{v}} P\wedge\mathbf{v}/c, \quad \gamma_b = \gamma_{\mathbf{v}}, \quad (44)$$

$$i_P \circ L_b^* = \gamma_{\mathbf{v}}\, i_{P+\mathbf{v}} - \left\{(e\circ g)\left(P + \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\mathbf{v}\right)\right\} \circ i_b. \quad (45)$$

Lemma 13. applied for the particular bivector (44), where $\mathbf{v}$ is the reciprocal *ternary* velocity parameterizing the Lorentz boost, cf. with (20), gives the bases-free derivation of the Lorentz transformation of the electric and magnetic field, (35),(36). We left the details to the reader.

7. Electric and Magnetic Fields without Lorentz Group

14. Assumption (Absolute electromagnetic field). In categorical kinematics it is postulated that the closed differential bi-form of electromagnetic field F, $dF = 0$, is observer-independent, independent of *any* observer, inertial or not inertial, *i.e.* F is absolute, [Kocik 1997; Cruz & Oziewicz 2003].

The concept of observer-independence in categorical kinematics is beyond the Lorentz-invariance in special relativity where the invariance is supposed with respect to transformations among *inertial* observers only.

In Einstein's special relativity the electric field is the strange concept: Lorentz-covariant? in what meaning? In categorical kinematics the electric and magnetic fields are coordinate-free, basis free, however, these fields are observer-dependent, *i.e.*, these fields are concomitant of a material observer-tensor field.

An observer, an endomorphism field $p \in \text{End}\,(\operatorname{der}\mathcal{F})$, extends to derivations of the tensor and Grassmann algebras, $p^* \in \operatorname{der}\{(\operatorname{der}\mathcal{F})^{*\wedge}\}$ [Cruz Guzman and Oziewicz 2003,2006]. We remaind that for a vector field Q, i_Q is a graded derivation of the Grassmann algebra of the differential forms, and, moreover, we have $i_Q g Q = -1$, and $(i_Q)^2 = 0$. A derivation i_Q is of grade -1, and derivations p and p^* are of zero grade. Therefore, $i_{\mathbf{v}} \in \text{der(Grass)}$, denotes the graded derivation of the Grassmann algebra, and, $i_{\mathbf{v}}E \equiv E\mathbf{v}$, is a scalar field given by the evaluation of the differential one-form E, on a vector field $\mathbf{v}$. Similarly, $i_{\mathbf{v}}B$, is a Pfaffian one-form. The commutator of derivations is again the derivation, and, $i_Q \circ q^* = i_Q$.

15. Definition (Electric and magnetic fields (Kocik 1997, Fecko 1997, Cruz and Oziewicz 2003)). The differential Pfaffian one-form of the electric field, $E(F,p)$, and differential bi-form of the magnetic field $B(F,p)$, relative to an observer p (*i.e.* measured by an observer p), are defined as follows

$$\begin{aligned} E(F,p) &\equiv i_P F, \\ B(F,p) &\equiv i_P\{(-gP) \wedge F\}. \end{aligned} \tag{46}$$

Present textbooks identify a differential one-form $i_P F \simeq P^\mu F_{\mu\nu}$, as the Lorentz force tangent to simultaneity of an observer P. Instead, already Tolman in [1918, page 114], and Eckart in 1940, identify correctly this differential one-form as a definition of a matter-dependent electric field.

Definitions (46) allows to relate observations of the electric and magnetic fields made by two observers in mutual relative motion. We are going to show the coordinate-free derivation of the transformations of the electric and magnetic fields within categorical kinematics, with binary velocity-isomorphism that not need to be constant. Ernst Mach was the first insisting that there must be transformations among observers that are related by *non*-constant relative velocity. We will show that categorical kinematics gives slightly different transformation of the electric and magnetic fields for a moving observer, as compared with special relativity with Lorentz transformations.

In what follows, and in particular in the next two Theorems 16. and 21., the absolute electromagnetic field F, is suppressed, and $\mathbf{v}/c \equiv \varpi_g(P,Q)$ is a binary (internal) velocity of an observer q relative to an observer p. This relative velocity not need to be constant, and the reference systems, p & q, need not to be inertial.

16. Theorem (Categorical kinematics). *Let $E(q)$ and $B(q)$ be the differential forms of electric and magnetic fields measured by an observer q. Let $E(p)$ and $B(p)$ be the differential forms of electric and magnetic fields measured by an observer p. Then, these physical fields are related by means of the following non-Lorentz expressions*

$$\begin{aligned} E(q) = \gamma_{\mathbf{v}}\left\{E(p) + i_{\mathbf{v}/c}B(p)\right\} \\ + \gamma_{\mathbf{v}}\left\{i_{\mathbf{v}/c}E(p)\right\} gP, \end{aligned} \tag{47}$$

$$\begin{aligned} B(q) = B(p) &+ (g\mathbf{v}/c) \wedge E(p) \\ &+ (gQ) \wedge \left\{ i_{\mathbf{v}/c}B(p) + \left(\gamma_{\mathbf{v}} - \frac{1}{\gamma_{\mathbf{v}}}\right) E(p) \right\} \\ &+ (\gamma_{\mathbf{v}}/c^2)\{i_{\mathbf{v}}E(p)\}g(\mathbf{v} \wedge P). \end{aligned} \tag{48}$$

Proof. The following inference does not need neither coordinates, nor basis, nor isometric Lorentz transformation,

$$i_{\mathbf{v}}F = i_{\mathbf{v}}B(p) + \{E(p)\mathbf{v}\}gP, \tag{49}$$

$$\begin{aligned} E(q) \equiv i_Q F &= \gamma_{\mathbf{v}}\left\{E(p) + i_{\mathbf{v}/c}F\right\} \\ &= \gamma_{\mathbf{v}}\left\{E(p) + i_{\mathbf{v}/c}B(p)\right\} + \gamma_{\mathbf{v}}\{E(p)\mathbf{v}\}gP. \end{aligned}$$

■

17. Lemma. *Let the differential one-form of the electric field, $E(p)$, the two-form of magnetic field $B(p)$, and a velocity vector field $\mathbf{v}/c \equiv \varpi_g(P,Q)$ of a body q relative to an observer p, are all measured by an observer p. This imply*

$$\begin{aligned} p\,\mathbf{v} \equiv 0, \; i_P E(p) \equiv 0, \; i_P B(p) \equiv 0, \\ q^*E(p) = -\gamma_{\mathbf{v}}\{i_{\mathbf{v}/c}E(p)\}\, gQ, \\ q^*B(p) = +\gamma_{\mathbf{v}}\{i_{\mathbf{v}/c}B(p)\} \wedge (gQ), \\ q^* i_{\mathbf{v}}B(p) = 0. \end{aligned} \tag{50}$$

7.1. Magnetic Field Orientation-Dependent

Most of textbooks of electromagnetics, wish to avoid Grassmann algebra of multivectors and multiforms, and therefore the magnetic field is defined artificially, in the framework of the Gibbs vector calculus of the right-hand-rule, as the pseudovector field, $\mathbf{B} \in \text{der}\,\mathcal{F}$, or as the differential pseudo-*one*-form $\mathbf{B} \equiv g\mathbf{B} \in (\text{der}\,\mathcal{F})^*$, Josiah Willard Gibbs admits that was not able to understand Hermann Grassmann. The price is that the pseudovector field $\mathbf{B}$, is orientation-dependent.

In dimension three, the orientation-dependent binary cross product of vectors, the vector product, $\vec{a} \times \vec{b}$, was invented by Clifford, popularized by Heaviside's *Electromagnetic Theory* [1893], and by Gibbs's *Vector Analysis* [1901]. Eckmann in 1942, considered, in an arbitrary dimension, the *multi*-ary generalization. Plebański with Przanowski [1988] introduced augmented quaternion-like algebra of para-vectors, and, in terms of this algebra, defined the binary cross product of vectors in arbitrary dimension. However, the dependence of these binary cross products on orientation is lost, or it is not explicit.

Here, we define the orientation-dependent binary cross product of multivectors, for dimension ≥ 3.

18. Definition (Gibbs's binary cross '$\times$' operation). A star, $*_g$, denotes the orientation-dependent Hodge star map. Let, $\dim_{\mathcal{F}}(\operatorname{der}\mathcal{F}) = n \in \mathbb{N}$, be spacetime dimension, $\operatorname{grade} A = \operatorname{grade} B = k \in \mathbb{N}$, and let, $\operatorname{grade} C = n - 3k$. The binary cross product, $\times_C$, on k-vectors is orientation-dependent and C-dependent, and is defined as follows

$$A \times_C B \equiv *_g(A \wedge B \wedge C). \tag{51}$$

Evidently, the same definition apply for differential multiform. This *binary* cross product does *not* exists for dimension ≤ 2. It is well defined for one-vectors for $3 \leq \dim$, for bi-vectors, for $6 \leq \dim$, etc. In particular, for $\dim = 3$, the cross depends on a scalar field, $\operatorname{grade} C = 0$, and for $\dim = 4$, the cross $\times_C$ depends on the choice of the auxiliary vector field C. Therefore in fact, it is a ternary operation, and not binary. This important vector-field-dependence of the cross in *fourth* dimension, is not realized (or suppressed), when writing four Maxwell differential equations and Lorentz transformations of the electric and magnetic fields.

In order to compare transformations in categorical kinematics with Lorentz transformations (35)-(36), we need temporarily also the artificial orientation-dependent magnetic pseudovector field, $\mathbf{B}$, in spite of our deep conviction that the physical measurable magnetic field, in fact, is orientation-free.

19. Definition (Magnetic field orientation-dependent). The magnetic field as the pseudovector field, $\mathbf{B} = g^{-1}\mathcal{B} \in \operatorname{der}\mathcal{F}$, and as a pseudo-one-form $\mathcal{B} = g\mathbf{B} \in (\operatorname{der}\mathcal{F})^*$, is orientation-dependent,

$$\mathcal{B}(p) \equiv \mathcal{B}(*_g F, p) = i_P *_g F.$$

20. Lemma. *The differential forms of the magnetic field are related as follows,*

$$\begin{aligned}(\det g)\, B(p) &= i_P *_g \mathcal{B}(p),\\ \mathcal{B}(p) &= i_P *_g B(p).\end{aligned} \tag{52}$$

Proof. On 2-forms the following identity holds, $(i_P \circ *_g)^2 = (\det g)(\mathrm{id} - p^*)$. ∎

21. Theorem (Categorical kinematics). *Let,* $\mathbf{E}(q) \equiv g^{-1}E(q)$, *and* $\mathbf{B}(q)$, *be the electric vector field and orientation-dependent magnetic pseudo-vector field measured by an observer* q. *Let,* $\mathbf{E}(p) \equiv g^{-1}E(p)$, *and* $\mathbf{B}(p)$, *be the electric and magnetic vector fields as measured by an observer* p. *Then, these fields are related by means of the following non-Lorentz expressions,*

$$\mathbf{E}(q) = \gamma_{\mathbf{v}} \left\{ \mathbf{E}(p) + (\det g)^{-1} \frac{\mathbf{v}}{c} \times_P \mathbf{B}(p) \right\} + \gamma_{\mathbf{v}} \left\{ \frac{\mathbf{v}}{c} \cdot \mathbf{E}(p) \right\} P, \tag{53}$$

$$\mathbf{B}(q) = \gamma_{\mathbf{v}} \left\{ \mathbf{B}(p) + \frac{\mathbf{v}}{c} \times_P \mathbf{E}(p) \right\} + \gamma_{\mathbf{v}} \left\{ \frac{\mathbf{v}}{c} \cdot \mathbf{B}(p) \right\} P. \tag{54}$$

In particular, the not Lorentz transformations (53),(54), imply the expected compatibilities

$$\begin{aligned}\mathbf{v}^{-1} \cdot \mathbf{E}(q) &= -\mathbf{v} \cdot \mathbf{E}(p),\\ \mathbf{v}^{-1} \cdot \mathbf{B}(q) &= -\mathbf{v} \cdot \mathbf{B}(p).\end{aligned} \tag{55}$$

Proof. The following expressions are a useful hint.

$$i_Q \circ *_g = *_g \circ e_{gQ} \cdot (-1)^{\mathrm{grade}}. \tag{56}$$

The next identity holds if restricted to Pfaffian forms only,

$$(\det g)^{-1} i_Q \circ *_g \circ i_P \circ *_g = \gamma_{\mathbf{v}} - \frac{1}{\gamma_{\mathbf{v}}} q^* - (g\mathbf{v}/c) \wedge i_Q. \tag{57}$$

This imply the following identity on 1- and 2-forms,

$$(i_P \circ *_g)^2 = (\det g)(\mathrm{id} - p^*). \tag{58}$$

In order to prove non-isometric transformations (53),(54), one should take into account that, $P \cdot \mathbf{E}(p) = P \cdot \mathbf{B}(p) = 0$, and moreover, one need to keep in the mind that the Gibbs cross product in four dimensions depends on extra vector (or covector), see Definition (51). The following identity holds,

$$(\det g)\, i_{\mathbf{v}} B(p) = (g\mathbf{v}) \times_{gP} \mathcal{B}. \tag{59}$$

∎

The only essential formal difference among the Lorentz transformation, and transformation given by categorical kinematics are terms containing the scalar products, $\mathbf{v} \cdot \mathbf{E}$ and $\mathbf{v} \cdot \mathbf{B}$. These terms in special relativity are always proportional to space-like relative velocity $\mathbf{v}$. In categorical kinematics these terms are time-like.

8. One Observer has many Clocks and Unique Proper-Time

A Pfaff differential form β, such that $\beta \wedge d\beta = 0$, is said to be Frobenius integrable.

22. Definition (Clock). Let β be a differential one-form, and P be a vector field. If β is transversal to P, *i.e.*, $\beta P \neq 0$, then β is said to be a P-clock (Frobenius integrable or not). Clock is metric-free.

P-clock is not unique. Let a differential one-form α be associated with the given vector field, *i.e.*, $\alpha P = 0$.

Then, $(\beta + \alpha)P \neq 0$, and $\beta + \alpha$, is again another P-clock (provided that $\beta+\alpha$ is Frobenius integrable). In particular, if a scalar field x is an integral for P, *i.e.*, $Px = 0$, and if $\beta = dt$, then, $t + x$ is again a P-clock, $P(t + x) \neq 0$, [Eckart 1940, page 920, formula (15)].

In the present paper, for the given time-like vector field P, we consider the following differential Pfaffian one-forms to be conceptually *independent*:

Metric-independent clock. The P-transversal differential form β, with $\beta P \neq 0$ (Frobenius integrable, or not integrable). This is mathematical *not*-unique P-clock. No condition on $\ker\beta$. P-clock is metric-independent and simultaneity-independent.

Metric-dependent proper-time. The differential Pfaff form, gP/P^2, with $(gP/P^2)P = 1$, is given uniquely by Einstein's gravitational potential g. This differential Pfaff form of the observer, introduced by Minkowski in 1908, gives the *unique empirical simultaneity*, provided is Frobenius integrable. In this case $\ker(gP)$ must be necessarily space-like. Llosa and Soler [2004] call this unique metric-dependent simultaneity, the Einstein proper-time. This proper-time-simultaneity was introduced by Minkowski in 1908.

In the present paper a *frame* is a synonym of a *basis* in a module. Basis is coordinate-free concept, known as the Cartan's 'moving frame'. Physical phenomenons are frame-free and coordinate-free.

The phrase *reference frame*, is used by other authors in many many different meanings. To give an overview of different understanding of the phrase *reference frame* would need a separate paper. For interesting reviews of the concepts of the reference frames and observers, different that ours, we refer among other to [Rodrigues & Capelas de Oliviera 2005], and to [Mitskievich 2006].

Few examples in chronological order.

Eckart [1940], in study of relativistic fluid, introduced absolute velocity of matter as normalized time-like vector-field.

Cattaneo [1958]. System (or fluid) of reference is associated to system of co-ordinates.

Vargas [1986]. A local frame is a free-falling elevator. There are several concepts, rest frame, preferred frame, inertial frame. Then, the reference frame is *relative* to the rest frame, and is defined as the pair: rigid body with ticking clocks.

Matolcsi [1993]. The material object like room and the car, are examples of observers [1993 §3]. This is the same as in the present paper. A reference frame (= a system) is a coordination of a time and of a space of an observer. In the present paper observer is coordinate-free and basis-free.

Kocik [1997]. As in [Eckart 1940]. An observer is a normalized time-like vector field, or equivalently a congruence of time-like curves. Following Eckart, this is called a perfect fluid by Misner, Thorne and Wheeler in 1973. The same in [Benn & Tucker 1987]. An individual observer is a single time-like curve. How an individual observer could be inertial?

Klioner and Soffel [1998]. A *reference system* is a synonym of a coordinate chart (why another name for well understood coordinate chart?). A reference frame is a materialization relative to coordinate chart, *i.e.*, a reference frame is coordinate-dependent. A coordinate frame, as used by Klioner and Soffel, is not a frame as used in the present paper. In the present paper a frame is a synonym of a mathematical basis, and this mathematical basis is coordinate-free.

Llosa and Soler [2004]. The reference-frame need space metric, therefore is not a basis, it is not a frame.

Rodrigues and Capelas de Oliviera, in a recent monograph [2005]. A frame is a synonym of a field of mathematical bases, Definition 508. Definition 288 say that a reference frame is a time-like vector field (or a congruence of time-like curves), as in [Eckart 1940]. Therefore, a reference frame is not a frame. Definition 281 say that an observer is a single time-like curve. How a curve could be inertial?

Pervushin [2005]. A reference frame is a three-dimensional coordinate basis with a watch.

This paper. A frame is a synonym of a mathematical basis. An observer is an idempotent $(1,1)$-tensor field, with time-like eigenvector. An observer (alias a reference *system*) is coordinate-free and basis-free, and must be massive (= time-like). In our kinematical groupoid-category: each object is some massive body represented by material idempotent field. Each morphism is a relative velocity represented by nilpotent field, Definition 7.. Therefore our kinematical groupoid-category is reduced to some operator algebra (of linear endomorphisms), with hope that this would be exactly an example of the Frobenius algebra, because massive bodies and relative velocities among them, all are given as $(1,1)$-tensor fields. More in [Cruz & Oziewicz 2006, Oziewicz 2007]. Note that the concept of a manifold is never used here explicitly. Instead, we deal exclusively with an associative algebra of scalar fields $\mathcal{F}$.

8.1. Minkowski's Proper-Time

If c is an embedding of a one-dimensional sub-manifold C, then the pull-back c^* is a morphism of algebras,

$$\begin{array}{ccc} C & \xrightarrow{\ c\ } & \text{space-time manifold } M \\ \mathcal{F}_C & \xleftarrow{\ c^*\ } & \text{algebra } \mathcal{F}_M. \end{array} \tag{60}$$

23. Definition (Integral curve). Let τ be coordinate on one-dimensional sub-manifold C, and $d/d\tau$ be a vector field on C. An embedding, $c : C \hookrightarrow$ manifold, is said to be integral curve for a vector field X, if the pull-back c^* intertwine the vector fields acting (as derivations) on tensor algebra and Grassmann algebra of differential forms. For arbitrary differential Pfaff form β, we have,

$$\frac{d}{d\tau} \circ c^* = c^* \circ X \quad \Longleftrightarrow \quad c^*\beta = \{(\beta X) \circ c\} d\tau. \tag{61}$$

In particular, $c^*(gX) = \{X^2 \circ c\} d\tau$.

Every *not* light-like vector field X, on pseudo-Riemannian manifold, posses the unique metric-dependent differential Pfaff form that define the metric-dependent scalar magnitude of the integral curves of X. For a time-like vector field X, this differential form is as follows,

$$\frac{-gX}{\sqrt{-g(X \otimes X)}}. \tag{62}$$

The g-dependent length of a curve 'c' is given by an integral,

$$\int_c \frac{-gX}{\sqrt{-g(X \otimes X)}} = \int_C c^* \frac{-gX}{\sqrt{-g(X \otimes X)}} = \int_C \sqrt{-X^2 \circ c}\, d\tau = \int_C d\tau, \quad \text{if } X^2 = -1. \tag{63}$$

Minkowski introduced this unique differential Pfaff form of the metric-dependent X-clock, *the X-proper-time*, $-gX/\sqrt{-g(X \otimes X)}$, implicitly in [1908, Appendix: Mechanics and relativity postulate, formula (3)] (considering this as the generalization of the Lorentz's *local time*).

$$\begin{aligned} &\text{Instead of:} \quad d\tau = c^* \frac{-gX}{\sqrt{-g(X \otimes X)}}, \\ &\text{Minkowski wrote:} \quad d\tau = \sqrt{-g}. \end{aligned} \tag{64}$$

Minkowski defined g-dependent X-proper-time $d\tau$, as the exact Pfaffian differential form, in terms of the integral curve 'c' of the time-like vector field X. The Frobenius-integrable, Minkowski's P-proper-time differential form, gP/P^2, for $P^2 < 0$, we identify with the unique empirical simultaneity of the massive body $p = P \otimes (gP/P^2)$.

Minkowski in 1908 wrote: '(space-like) vector $\mathbf{v}/c$, uniquely define (time-like) normalized vector Q, and vice versa', [Minkowski 1908, §4: Special Lorentz transformations, before formula (20)]. This must be understood that every *pair* of normalized time-like vectors, P and Q, determine the unique binary relative velocity-morphism, $\mathbf{v}/c = Q/\gamma_{\mathbf{v}} - P$, such that $P \cdot \mathbf{v} = 0$. This is exactly Definition 3., and formula (15). However, it is misleading to interpret Definition 3. as the 'special Lorentz transformations', as did Minkowski. This interpretation is not correct, because the velocity-morphism is not reciprocal, and therefore can not parameterize the isometric Lorentz boost.

The given space-like velocity, $\mathbf{v}$, can be the relative velocity among *many* different pairs of massive bodies. There is no one-to-one correspondence among pairs of massive (time-like) bodies and relative (space-like) velocities, $Q/\gamma_{\mathbf{v}} - P = Q'/\gamma_{\mathbf{v}} - P' = \ldots$ This non-uniqueness follows from the fact that the following system of two algebraic equations in dimension four, for the given space-like vector $\mathbf{v}$, $P \cdot \mathbf{v} = 0$ and $P^2 = -1$, has a two-parametric family of solutions. For the given space-like $\mathbf{v}$, categorical kinematics gives many different inverses $\mathbf{v}^{-1}$, observer-dependent, given explicitly by formula (23), $\mathbf{v}^{-1} = \mathbf{v}^{-1}(\mathbf{v}, P)$.

In Minkowski's formula (20) in [1908 §4], the relative velocity $\mathbf{v}$, clearly is the *binary* velocity, as our Definition (15),

$$-gQ = \gamma_{\mathbf{v}} \left(-gP - g\frac{\mathbf{v}}{c}\right), \tag{65}$$

$$\begin{aligned} d\tau = \{\sqrt{-g}\}_{\text{Minkowski}} &= c^*(-gQ) \\ &= (\gamma \circ c) \left\{c^*(-gP) - c^* \left(g\frac{\mathbf{v}}{c}\right)\right\}. \end{aligned} \tag{66}$$

Here, c, is an integral curve of the vector field Q, and not of the vector field P. Moreover,

$$\begin{aligned} g = -(dt) \otimes (dt) &+ \frac{1}{c^2}(dx) \otimes (dx) \\ \xrightarrow{\ c^*\ } \quad &\{d(t \circ c)\}^2 + \frac{1}{c^2}\{d(x \circ c)\}^2, \\ d(x \circ c) &= w\, d(t \circ c), \\ d\,\mathrm{id} &= \sqrt{1 - \frac{w^2}{c^2}}\, d(t \circ c). \end{aligned} \tag{67}$$

9. Adopted co-Frame and Metric Tensor

In the rest of this paper, for simplicity, we consider 2-dimensional space-time with signature $(-, +)$ only.

24. Definition (Adopted co-frame). A co-frame is a mathematical basis in the module of the differential Pfaff forms, $\beta \wedge \alpha \neq 0$. A co-frame is said to be *adopted* for a vector field P, if $\beta P = 1$ and $\alpha P = 0$. Briefly, a co-frame adopted for a vector field P, is said to be a P-co-frame.

When considering several different vector fields, we will denote P-co-frame using index, α^p and β^p. A dual P-frame of a vector fields is, $P \wedge X_p \neq 0$, where $\alpha^p X_p \equiv 1$ and $\beta^p X_p \equiv 0$. A P-frame, $P \wedge X_p \neq 0$, need not to be orthogonal.

In dimension two, the differential form α is unique up the scalar field (up to integrating factor). For example if $\alpha = dx$, then a scalar field x is an integral for P. If $a \in \mathbb{R}$ is a non zero constant, then, ax is again integral for the same vector field P, $P(ax) = 0$. However the change of coordinates, from x to ax, must not be interpreted as the *material* rod contraction or expansion.

In dimension two, there is a two-parametric family of adopted co-frames for each massive body,

$$IGL1 \simeq \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset GL2, \qquad \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \longrightarrow \begin{pmatrix} \beta + f\alpha \\ g\alpha \end{pmatrix}. \tag{68}$$

In dimension $n \in \mathbb{N}$, the adopted co-frame is given up to the action of the following inhomogeneous $GL(n-1)$ subgroup,

$$\begin{pmatrix} 1 & * \\ 0 & GL(n-1) \end{pmatrix} \quad \subset \quad GLn. \tag{69}$$

An observer in the present paper is a tensor $(1,1)$-field p, and therefore, it is basis-free and coordinate-free. In particular, time-like eigenvector P, $pP = P$, and the Minkowski simultaneity differential form, gP/P^2, are both frame-free and coordinate-free. Any *non-inertial* observer p, (self-acceleration is related to the differential bi-form $d(gP/P^2)$), can be presented in his P-co-frame, in the following way, where we must appreciate the difference among the unique Einstein-Minkowski physical simultaneity differential one-form, gP/P^2, and the many mathematical P-clocks, $\beta, \beta', \ldots$, that does not need to be necessarily Frobenius integrable, see Definition 22.,

$$p \equiv \frac{P \otimes (gP)}{P^2} \quad \text{with} \quad gP/P^2 = \beta + f\alpha = \beta' + f'\alpha = \ldots \tag{70}$$

$$\ker(gP) = \operatorname{span}\{X + (P \cdot X)P\} \ \text{must be space-like} \iff 0 < X^2 + (P \cdot X)^2. \tag{71}$$

The metric tensors, g (in units s^2), and g^{-1} (in s^{-2}), need not to be diagonal in P-co-frame and in dual P-frame. In particular we do not suppose that the curvature tensor field must vanish. This imply two-parametric family of metric tensors. Consider the pair of scalar fields

$$m \equiv X^2 \equiv g(X \otimes X), \qquad n \equiv g(P \otimes X) = g(X \otimes P). \tag{72}$$

The metric field in P-co-frame is as follows,

$$gP = -\beta + n\alpha \quad \Longrightarrow \quad g(\beta - n\alpha) = -P, \tag{73}$$

$$gX = n\beta + m\alpha, \tag{74}$$

$$g\beta = -P + \frac{n}{m+n^2}(X + nP), \tag{75}$$

$$g\alpha = \frac{1}{m+n^2}(X + nP) \tag{76}$$

Consider the following volume form

$$\eta \equiv \beta \wedge \alpha, \qquad \eta(P \wedge X) = 1. \tag{77}$$

With respect to this volume form η, one can calculate the η-dependent determinant of g, *i.e.*, the scalar field,

$$\det\, g \equiv *\eta \equiv \det_\eta g \equiv \eta(g\eta), \tag{78}$$

$$g\eta = g(\beta \wedge \alpha) = -\frac{1}{m+n^2}\, P \wedge X \quad \Longrightarrow \quad \det g = -\frac{1}{m+n^2}, \tag{79}$$

$$\det g < 0 \quad \Longleftrightarrow \quad 0 < m + n^2. \tag{80}$$

This can be summarized as

$$\begin{aligned} P \cdot P &= -1, & \beta \cdot \beta &= m \det g, \\ X \cdot X &= m & \alpha \cdot \alpha &= -\det g, \\ P \cdot X &= n & \beta \cdot \alpha &= -n \det g. \end{aligned} \tag{81}$$

$$g = -\beta \otimes \beta + m\, \alpha \otimes \alpha + n(\alpha \otimes \beta + \beta \otimes \alpha), \tag{82}$$

$$g^{-1} = (\det g)\{m\, P \otimes P - X \otimes X - n(P \otimes X + X \otimes P)\}, \tag{83}$$

$$-gP = \beta - n\alpha. \tag{84}$$

10. Transformations of Adopted co-Frames

The aim of this Section is the derivation the transformations among adopted co-frames, from $\{\beta^p, \alpha^p\}$ to $\{\beta^q, \alpha^q\}$, within the categorical kinematics and without Lorentz transformations. The co-frames are adopted for a pair of not necessarily inertial observers in mutual motion. The adopted-frame transformations are derived in terms of the binary relative velocity-morphism only (not-isometric and not-reciprocal). Therefore these transformations differs from Poincaré and Lorentz isometry.

In this Section we are showing that the family of transformations of the mathematical clocks of monads, include, among other, Stefan Marinov's transformation with the absolute mathematical clock [Marinov 1974, 1977], as well as Tangherlini transformation.

We must distinguish among frame-free and coordinate-free concepts, and the concepts that depends explicitly on the choice of the co-frame, and, in particular, a choice-dependent coordinates. All tensor fields, p and q, eigenvector fields P and Q, simultaneity differential forms, $-gP$ and $-gQ$, are frame-free. The change of co-frames can not change neither observer nor observed body, nor a differential bi-form of the relativity of simultaneity $g(P \wedge Q)$. The relativity of simultaneity is the physical concept and must be independent on the mathematical choice of co-frames and coordinates.

25. Clarification (Which time is relative? and when?). Every observer, a material tensor field p, posses the unique empirical simultaneity differential form, gP/P^2, and many mathematical P-clocks $\{\beta^p\}$, where $\ker \beta^p$ is isochronous for an observer p. The simultaneity of an observed body q, is given by a kernel of empirical differential form, $\ker(gQ)$, and not by a mathematical clock β^q. We need avoid presupposing an artificial identification of the choice-free simultaneity, gP/P^2, with the choice-dependent mathematical clock β.

26. Assumption (Two-body system). All considerations are valid for arbitrary non-inertial observers, and for *variable* binary relative velocity-morphism $\mathbf{v} \equiv c\,\varpi_g(P,Q)$. Consider two-body system,

$$P^2 = Q^2 = -1 \tag{85}$$

$$p = P \otimes (-gP), \quad q = Q \otimes (-gQ), \tag{86}$$

$$\frac{\mathbf{v}}{c} \equiv \varpi(P,Q) = \frac{Q}{\gamma_{\mathbf{v}}} - P \in \ker gP, \tag{87}$$

$$0 \neq \gamma_{\mathbf{v}} \equiv -P \cdot Q, \tag{88}$$

$$Q \wedge P = \gamma_{\mathbf{v}} \frac{\mathbf{v}}{c} \wedge P = \gamma_{\mathbf{v}}\, Q \wedge \frac{\mathbf{v}^{-1}}{c}. \tag{89}$$

We denote by β^q, α^q, an adopted Q-co-frame, *i.e.*, $\alpha^q Q \equiv 0, \beta^q Q \equiv 1$. Analogously β^p, α^p, is an adopted P-co-frame. A differential form β^p is not-unique P-clock,

$$P \otimes \beta^p + X_p \otimes \alpha^p = \mathrm{id}, \tag{90}$$

$$\mathbf{v} = (\beta^p \mathbf{v})P + (\alpha^p \mathbf{v})X_p = (\beta^q \mathbf{v})Q + (\alpha^q \mathbf{v})X_q, \tag{91}$$

$$\mathbf{v} \neq 0 \iff \alpha^p \mathbf{v} \neq 0 \quad \& \quad \alpha^q \mathbf{v} \neq 0. \tag{92}$$

The Minkowski differential form, $-gP \in (\operatorname{der} \mathcal{F})^*$, is the unique empirical simultaneity differential form for an observer p. The concept of the relative velocity is frame-independent, but it is metric-dependent, needs Einstein-Minkowski simultaneity. We are going to show that the length/rod-contraction or extension, and mathematical clock-dilation and clock-retardation, in categorical kinematics are frame-dependent. Being frame-dependent are not material and not biological respectively. In contrast, the dilation of simultaneity is frame-free. Being frame-free has the objective physical meaning.

When $\mathbf{v} \equiv c\varpi_g(P,Q)$, is a binary velocity, then his inverse, $\mathbf{v}^{-1} \equiv c\,\varpi(Q,P) \neq -\mathbf{v}$, is observer-dependent, and it is not absolute, Theorem 6.,

$$(-gP)\frac{\mathbf{v}^{-1}}{c} = -\gamma_{\mathbf{v}} + \frac{1}{\gamma_{\mathbf{v}}}, \quad (-gP)\mathbf{v} = 0. \tag{93}$$

Therefore, within the categorical kinematics, the inverse transformation in terms of the same velocity $\mathbf{v}$, is more complicated, because the binary velocity $\mathbf{v}$ is *not* measured by an observer Q.

27. Definition (Scalar fields). The following six frame-*dependent* scalar fields are defined as follows.

$$\begin{aligned} a &\equiv \alpha^q X_p, \quad \tilde{a} \equiv \alpha^p X_q, \\ h &\equiv \beta^q X_p, \quad \tilde{h} \equiv \beta^p X_q, \\ j &\equiv \beta^q P = \frac{1}{\gamma_{\mathbf{v}}} - \beta^q \frac{\mathbf{v}}{c}, \\ \tilde{j} &\equiv \beta^p Q = \gamma_{\mathbf{v}} \left(1 + \beta^p \frac{\mathbf{v}}{c}\right). \end{aligned} \tag{94}$$

The notation in the above Definition we made in analogy for Robertson's in 1949, who introduced the test-theory with 'the most general coordinate transformations consistent with ...'. The Robertson coordinate transformations are used extensively by Vargas [Vargas 1984, 1986 pages 1005, 1091]. For example, an analogy of our $\tilde{a}$ is denoted by Vargas by A, our $\tilde{h}$ by C, and our $\tilde{j}$ by B, see [Vargas 1986, page 1091]. However our kinematical conditions (93)-(**??**) are disregarded by Vargas. For example, in [Vargas 1986, page 1093], $\mathbf{v}^{-1} = -\mathbf{v}$.

28. Theorem. *The following identities must hold,*

$$\begin{aligned} \alpha^q &= a\alpha^p - \left(\alpha^q \frac{\mathbf{v}}{c}\right)\beta^p, \\ \alpha^p &= \tilde{a}\alpha^q + \gamma_{\mathbf{v}}\left(\alpha^p \frac{\mathbf{v}}{c}\right)\beta^q, \\ \beta^q &= j\beta^p + h\alpha^p, \quad \beta^p = \tilde{j}\beta^q + \tilde{h}\alpha^q, \end{aligned} \tag{95}$$

$$\left\{aj + h\alpha^q \frac{\mathbf{v}}{c}\right\}\left\{\tilde{a}\tilde{j} - \gamma_{\mathbf{v}}\tilde{h}\alpha^p \frac{\mathbf{v}}{c}\right\} = 1, \tag{96}$$

$$\beta^q \wedge \beta^p = h\,\alpha^p \wedge \beta^p = -\tilde{h}\,\alpha^q \wedge \beta^q, \tag{97}$$

$$\alpha^q \wedge \alpha^p = \left(\alpha^q \frac{\mathbf{v}}{c}\right)\alpha^p \wedge \beta^p = \gamma_{\mathbf{v}}\left(\alpha^p \frac{\mathbf{v}}{c}\right)\alpha^q \wedge \beta^q, \tag{98}$$

$$(\alpha^q \wedge \beta^q)\left(\frac{\mathbf{v}}{c} \wedge X_p\right) + \frac{a}{\gamma_{\mathbf{v}}} = aj + h\alpha^q \frac{\mathbf{v}}{c} \tag{99}$$

$$\begin{aligned} j + \left(\alpha^p \frac{\mathbf{v}}{c}\right)h &= \frac{1}{\gamma_{\mathbf{v}}} - \left(\beta^p \frac{\mathbf{v}}{c}\right)j, \\ \tilde{j} &= \gamma_{\mathbf{v}}\left\{1 + \left(\alpha^q \frac{\mathbf{v}}{c}\right)\tilde{h} + \left(\beta^q \frac{\mathbf{v}}{c}\right)\tilde{j}\right\}. \end{aligned} \tag{100}$$

Hint. We need to use all our assumptions (86)–(**??**), that are analogous to the test-theory by Vargas [Vargas 1986, page 1091, formula (3)]. For example, from definition of the binary velocity-isomorphism, (87)–(89), we have,

$$Q = \gamma_{\mathbf{v}}(P + \mathbf{v}/c), \tag{101}$$

$$\alpha^q P = -\alpha^q \mathbf{v}/c, \tag{102}$$

$$\alpha^p Q = \gamma_{\mathbf{v}} \alpha^p \mathbf{v}/c = -\alpha^p \mathbf{v}^{-1}/c. \tag{103}$$

□

Note that here we do not use Lorentz transformation, our transformation is not an isometry, because among other, we used kinematical constraints (24),(93).

Robertson's boost of mathematical frames [Robertson 1949], depends on three scalars a, h, j. Vargas shown, that the Ives and Stillwell optical experiment [Ives and Stillwell 1938; Vargas 1984, 1986, page 1097], agrees with the transformation of the proper times (95)–(100), if these clocks are given by Minkowski simultaneity, *i.e.* if, in an identity (100), $\beta^p\, \mathbf{v} = 0 \in \mathcal{F}$.

The frame-dependent scalar fields, a and $\tilde{a}$, are responsible for the lenght/rod contraction/extension. This contraction is *non*-physical, non-material phenomenon, because depends on the convenient choice of the mathematical frame. The categorical kinematics, with *binary* relative velocity, does not predicts the lenght/rod *material* contraction, because of the mathematical freedom in the choice of the adopted co-frames.

For the scalar fields, $a, h, j, \tilde{a}, \tilde{h}, \tilde{j}$, subject to the identities given by Theorem 28., identities (95)-(100), the co-frame $\{\beta^q, \alpha^q\}$, is Q-co-frame for a moved body Q. For example, one can chose, $a = 1$ and $\tilde{a} = 1$, *i.e.*, no rod contraction.

This conclusion differs from Einstein's isometric more restrictive formulation in terms of the *ternary* velocities, and can be related with conclusion made by Jefimenko [1998]. Our conclusion again rise the long standing problem of the role of mathematical frames and coordinates in physics, compare, e.g., with discussion in [Vargas 1986, page 1112]. Does Nature like the preferred frames and preferred coordinate-systems? like the Fock's harmonic coordinates?

Larmor's dilation and Einstein's isometry. The condition, $h\tilde{h} \neq 0$, is frame-dependent (not absolute), see identities (97), and it is equivalent to relativity of the mathematical clocks, $\beta^q \wedge \beta^p \neq 0$. However, the simultaneity must be relative, and is basis-free and coordinate-free,

$$\mathbf{v} \equiv \varpi(P, Q) \neq 0 \quad \Longleftrightarrow \quad (-gP) \wedge (-gQ) = g(P \wedge Q) \neq 0, \tag{104}$$

$$Q \wedge P = -\gamma_{\mathbf{v}} \frac{\mathbf{v}}{c} \wedge P. \tag{105}$$

The Larmor's dilation of simultaneity [Larmor 1897, 1900], and Einstein's transformation need empirical simultaneity, $\beta^p = -gP$ and $\beta^q = -gQ$. This imply $j = \tilde{j} = \gamma$. The simultaneity of a P-co-frame is, $(-gP) \wedge \alpha^p \neq 0$, and then, a dual P-frame is g-orthogonal, $P \wedge X^p \neq 0$ and $P \cdot X_p = 0$. The Einstein-Minkowski simultaneity must be delated,

$$-gQ = \gamma_{\mathbf{v}} \left(-gP - g\frac{\mathbf{v}}{c} \right),$$

$$-gP = \gamma_{\mathbf{v}} \left(-gQ - g\frac{\mathbf{v}^{-1}}{c} \right),$$

$$g\mathbf{v} = (\alpha^p \mathbf{v}) g X_p.$$

This is the same as the Lorentz transformation of simultaneity, however we are *not* obliged for the choice, $\alpha^p = gX_p$.

Marinov's absolute clock. When $h = 0 = \tilde{h}$, a clock is absolute, $\beta^q \wedge \beta^p = 0$. This implies, using identity (96), that, $a\tilde{a} = 1$ and $j\tilde{j} = 1$. Let, in particular, $\beta^p \mathbf{v} = 0 = \beta^q \mathbf{v}$. Then, $j = 1/\gamma$ and $\tilde{j} = \gamma$, *i.e.*, clock is retarded, $\beta^q = (1/\gamma)\beta^p$. This clock retardation is not biological, because is frame-dependent. The Marinov's transformation of adopted frames (and Tangherlini transformation), means that there is an absolute mathematical clock, however this does *not* imply that simultaneity is absolute. Therefore Marinov's transformation has nothing to do with the Galilean absolute simultaneity. Compare with [Marinov 1974,1977; Vargas 1984, page 646; Vargas and Torr 1986 p. 1115]. Vargas claim that the Marinov transformation is an alternative to isometric special relativity. In relation to this we like to stress the following facts

- We deduced the Marinov transformation of non-unique adopted mathematical frames, see (68)-(69), from non-isometric transformation of observers.
- Transformation of adopted frames is mathematical convenience only.

Clocks without retardation? Question: can we choose the mathematical clocks, β^p and β^q, in such way that there would be neither clock-dilation and nor clock-retardation? This means, can we chose in Theorem 28. $j = 1 = \tilde{j}$?

29. Corollary (Galileo Galilei 1632, space is relative). By definition, $\ker \alpha^p$, is a set of locations/points in a space of an observer p, $\alpha^p P = 0$.

Analogously, $\ker\alpha^q$, is a space of a body q, $\alpha^q Q\equiv 0$. For not vanishing relative velocity $\mathbf{v}\neq 0$, relativity of spaces is obligatory. The identity (98) imply

$$\alpha^q\wedge\alpha^p=0 \quad\Longrightarrow \quad \mathbf{v} \text{ must be time-like, a contradiction,} \tag{106}$$

$$\mathbf{v}\neq 0 \quad\Longleftrightarrow\quad \alpha^q\wedge\alpha^p = -\left(\alpha^q\mathbf{v}/c\right)\,\beta^p\wedge\alpha^p\neq 0. \tag{107}$$

For review of the concepts of a space, we refer to [Jammer 1954].

In adopted non-unique mathematical coordinates, $P=\partial_t$, $Q=\partial_{t'}$ and, $-gQ=dt'$.

11. Comments on some Related Publications

For the test-theory of relativity we refer to [Robertson 1949; Vargas 1984,...,2000; Anderson, Vetharaniam and Stedman 1998]. The test-theory consider general linear GL-transformations, the velocity-dependent transformations of mathematical frames. The categorical groupoid kinematics, with our Corollary 8., is not included in these 'covering theories of relativistic physics'. The groupoid kinematics is defined by means of the transformations among massive, time-like observers-idempotents only. The Robertson GL-covering theory assume the linear relation among relative velocity $\mathbf{v}$, and his inverse $\mathbf{v}^{-1}$, *i.e.*, it is assumed ab initio implicitly the vanishing of the following bivector, $\mathbf{v}^{-1}\wedge\mathbf{v}=0$, and therefore also $\left(\mathbf{v}^{-1}\wedge\mathbf{v}\right)^2=0$. See, for example, [Vargas 1986, page 1091, formula (4)]. It is the crucial property of the groupoid structure of the categorical kinematics that this bivector can *not* vanish.

Another distinguished property of the categorical kinematics is that the transformations among systems can not be always composed, the group structure is lost. In the categorical groupoid theory all systems/objects are on the same footing and there is no privileged observer.

Alladi Ramakrishnan in the years 1973–1883 published series of papers on special relativity. Ramakrishnan in 1978 introduced new concept in special relativity, namely the exterior and interior observers, and correspondingly, the interior relative velocity, v_i^j, of a particle j with respect to (interior) particle i, and the *exterior* ternary velocity, $v_k^j(i)$, that is the velocity of j with respect to k *as observed* by a particle i. Ramakrishnan *assumed* that both these velocities, interior and exterior, *are* reciprocal, $v_i^j=-v_j^i$. This means therefore, (according to Theorem 6.), that in fact, Ramakrishnan's 'interior' and 'exterior' velocities must be ternary (with hidden extra particle/observer). Therefore categorical theory of binary and ternary relative velocities is different from Ramakrishnan's theory. In categorical kinematics one can express every isometric ternary velocity as the 'subtraction' of binary velocities-morphisms (this will be published elsewhere).

12. Outline

Categorical kinematics is Lorentz-group-free, and gives well-defined center-of-mass for relativistic many-body system (*i.e.* exterior-observer-independent), as opposite to relativity with isometry-morphisms.

By categorical morphisms, the addition of binary-velocities-morphisms is associative, as opposite to non associative addition of exterior ternary isometric velocities [Oziewicz 2005; Page 2006].

The main conclusion is: observer-independence/dependence and the Lorentz/Poincaré-group-invariance/covariance are *different* concepts. The same statement holds in Newtonian physics: observer-independence is not be the same as the Galilean-group-invariance.

We suggest novel formulation of the many-body exterior-observer-independent relativistic dynamics without Lorentz/Poincare invariance, with necessarily violation of the third Newton's law.

The theory of the electromagnetic phenomena in the framework of the categorical kinematics differers as compared to Lorentz-group isometric relativity where a morphism among reference systems *must* be an isometry of an empty spacetime, Theorems 16. and 21..

Within the categorical kinematics the *inverse* of the relative velocity, $\mathbf{v}^{-1}$, must be interior-observer-dependent, $\mathbf{v}^{-1}$ is P-dependent, and no more absolute,

$$\left(\mathbf{v}^{-1}\wedge\mathbf{v}\right)^2=-\frac{\left(\gamma_{\mathbf{v}}^2-1\right)^3}{\gamma_{\mathbf{v}}^4}\,c^4, \tag{108}$$

and therefore, $\mathbf{v}^{-1}\neq-\mathbf{v}$. The Einstein's isometric exterior formulation needs, instead, the vanishing bivector, $\mathbf{v}^{-1}\wedge\mathbf{v}\equiv 0$.

Acknowledgment

Author is indebted to William Page (Kingston, Canada) for insightful observations, valuable discussions, and for not discouraging categorical kinematics. Thanks to David Cyganski (USA) for pointing the relevant references, the monograph by Tomás Matolcsi [1984, 1993], and publication by Bini, Carini and Jantzen [1995], among many other. I wish to thank to Bernard Jancewicz (Wroclaw University, Poland) who kindly read some partial drafts, made several comments, and corrected English, however is not responsible for the bad English in this final version.

Manuscript received March 22, 2006

References

[1] Anderson R., I. Vetharaniam, and G. E. Stedman, Conventionality of synchronisation, gauge dependence and test theories of relativity // Physics Reports – 1998. – V. 235. – P. 93–180.

[2] Bargmann V., Relativity // Reviews of Modern Physics – 1957. – V. 29, № 2. – P. 161–174.

[3] Bini Donato, Paolo Carini, and Robert T. Jantzen, Relative observer kinematics in general relativity // Classical and Quantum Gravity – 1995. – V. 12. – P. 2549–2563.

[4] Bolós V. J., Lightlike simultaneity, comoving observers, and distances in general relativity // Journal of Geometry and Physics – 2006. – V. 56. – P. 813–829.

– gr-qc/0501085, version 3. – April 25, 2005.

[5] Bolós V. J., Intrinsic definitions of 'relative velocity' in general relativity // arXiv:gr-qc/0506032 version 3 (24 November 2006)

[6] Brillouin Léon, Relativity Reexamined. – New York: Academic Press. – 1970.

[7] Cattaneo C., General relativity: relative standard mass, momentum, energy and gravitational field in a general system of reference // Il Nuovo Cimento – 1958. – V. X, № 2. – P. 318–337.

[8] Cruz Guzmán José de Jesús, and Zbigniew Oziewicz, Four Maxwell's equations for non-inertial observer // Bulletin de la Société de Sciences et des Lettres de Łodź – 2003. – V. LIII, Série: Recherches sur les Déformations. – V. XXXIX. – P. 107–140.

[9] Cruz Guzmán José de Jesús, and Zbigniew Oziewicz, Nijenhuis-Richardson algebra, and Frölicher-Nijenhuis Lie module, in: Lev Sabinin, Larissa Sbitneva and Ivan Shestakov, Editors, Non-Associative Algebra and Its Applications. – Boca Raton, London: Chapman & Hall/CRC Press. – 2006. Chapter 9. – P. 109–127.

[10] Cunningham E., The principle of relativity in electrodynamics and an extension there of // Proceedings of London Mathemtical Society – 1909. – V. 8 (February 8). – P. 77–98.

[11] Dvoeglazov Valeri V., and J. L. Quintanar González, A note on the Lorentz transformations for the photon, Foundations Physics Letters – April 2006.

[12] Eckart Carl, The thermodynamics of irreversible processes, Part III. Relativistic theory of simple fluid // Physical Review – 1940. – V. 58. – P. 919–924.

[13] Einstein Albert, Zur Electrodynamik bewegter Körper // Annalen der Physik (Leipzig) – 1905. – V. 17. – P. 891–921.

[14] Ernst Andreas, and Jong-Ping Hsu, First proposal of the universal speed of light by Voigt in 1887 // Chinese Journal of Physics – 2001. – V. 39, № 3. – P. 211–230.

[15] Fecko Marián, On 3 + 1 decomposition with respect to an observer field via differential forms // Journal of Mathematical Physics – 1997. – V. 38, № 9. – P. 4542–4560.

[16] FitzGerald G. F., The ether and the Earth's atmosphere, Science (USA) – 1889. – V. 13. – P. 390.

[17] Galileo Galilei, Dialogo ... sopra i due massimi sistemi del mondo: Tolemaico e Copernico (Dialog concerning the two chief world systems). Editor G. B. Landini. – Florence: – 1632.

[18] Gibbs Josiah Willard, Vector Analysis. – 1901.

[19] Gill Tepper L., and James Lindesay, // International Journal of Theoretical Physics – 1993. – V. 32. – P. 2087.

[20] Gottlieb Daniel Henry, Skew symmetric bundle maps on space-time // Contemporary Mathematics – 1997. arXiv q-alg/9603024.

[21] Grassmann Hermann, Lineale Ausdebnungslebre, ein neuer Zweig der Mathematik. – Leipzig: Druck und Verlag. – 1844. – Buenos Aires: – 1947.

[22] Hajra Sankar, and Antina Ghosh, Collapse of SRT 1: derivation of electrodynamics equations from the Maxwell field equations // Galilean Electrodynamnics – 2005. – V. 16, № 4. – P. 63–70.

[23] Harnett G., Metrics and dual operators // Journal of Mathematical Physics – 1991. – V. 32, № 1 – P. 84–91.

[24] Harnett G., The bivector Clifford algebra and the geometry of Hodge dual operators // Journal of Physics A: Mathematics and General – 1992. – V. 25. – P. 5649–5662.

[25] Heaviside Oliver, The electro-magnetic effects of a moving charge // The Electrician – 1888. – V. 22. – P. 147–148.

[26] Heaviside Oliver // Philosophical Magazine – 1889. – V. 27. – P. 324–339.

[27] Heaviside Oliver, Electromagnetic Theory. – 1893

[28] Hestenes David, Space-Time Algebra. – New York: Gordon and Breach. – 1966.

[29] Honig William M., Relativity of the metric // Foundations of Physics – 1977. – V. 7, № 7/8. – P. 549–572.

[30] Ivezić Tomislav // Foundation Physics Letters – 1999. V. 12. – P. 105. and P. 507.

[31] Ivezić Tomislav // physics/0007031, physics/0012048.

[32] Ivezić Tomislav, The 'true transformations relativity', analysis of the Michelson-Morley experiment // Physics Essay – 2001. – V. 14, № 3. physics/0101091.

[33] Ivezić Tomislav // Foundation of Physics – 2001. – V. 8. – P. 1139.

[34] Ivezić Tomislav, An invariant formulation of special relativity, or the 'true transformation relativity', and its comparison with experiments // Foundation Physics Letters – 2002. – V. 15. – P. 27. physics/0103026.

[35] Jackson John David, Classical Electrodynamics. – John Wiley & Sons New York. – 1962, 1975, 1999.

[36] Jammer M., Concepts of Space. – Harward Press. – 1954.

[37] Jancewicz Bernard, Multivectors and Clifford Algebra in Electrodynamics. – World Scientific, Singapore. – 1989.

[38] Jancewicz Bernard, Wielkości skierowane w elektrodynamice. – Uniwersytet Wrocławski. – 2000.

[39] Jefimenko Oleg D. On the experimental proofs of relativistic length contraction and time dilation // Zeitschrift für Naturforschung – 1998. – V. 53a. – P. 977–982.

[40] Klioner Sergei A., and Michael Soffel, Nonrotating astronomical relativistic reference frames // Astronomy and Astrophysics. – 1998. – V. 334. – P. 1123–1135.

[41] Kocik Jerzy, Relativistic observer and Maxwell's equations: an example of a non-principal Ehresmann connection // http://www.math.siu.edu/Kocik/ – 1997.

[42] Kotelnikov G. A., Universal Newton time in classical electrodynamics. Elements of physical interpretation // arXiv:physics/9802038 – 1998.

[43] Landau Lev Davidovich, and Eugeni M. Lifshitz, The Classical Theory of Fields. – Moscow: – 1973. Pergamon Oxford – 1975.

[44] Larmor Joseph Sir, Aether and Matter. Cambridge University Press – 1900.

[45] Lévy-Leblond J. // American Journal of Physics. – 1980. – V. 48. – P. 345.

[46] Llosa J., and D. Soler, Reference frames and rigid motions in relativity // Classical and Quantum Gravity. – 2004. – V. 21. – P. 3067–3094.

[47] Lorentz H. Antoon // Zittingsverslagen der Koninklijke Akademie Van Wetenschappen te Amsterdam. – 1882–1883. P. 74; Deel – 1904. – V. XII. – P. 986. Proceedings – V. 6. – P. 809.

[48] Marinov Stefan // Czechoslovak Journal of Physics – 1974. – V. 9. – P. 139.

[49] Marinov Stefan, Eppur si muove. – Bruxelles: Centre Belge de Documentacion Scientifique. – 1977. – Graz: East-West. – 1981.

[50] Matolcsi Tomás, A concept of Mathematical Physics, Models for Spacetime. – Budapest: Akadémiai Kiadó. – 1984.

[51] Matolcsi Tomás, Spacetime without reference frames. – Budapest: Akadémiai Kiadó. – 1993. – 1994.

[52] Matolcsi Tomás and Goher A. Spacetime without reference frames: An application to the velocity addition paradox // Studies in History and Philosophy of Modern Physics – 2001. – V. 32, № 1. – P. 83–99.

[53] Mattingly David, Modern tests of Lorentz invariance // Living Reviews in Relativity – 2005. – V. 8. – P. 5.

http://www.livingreviews.org/lrr-2005-5

[54] Minkowski Hermann, Die Grundlagen für die electromagnetischen Vorgänge in bewegten Körpen // Nachr. König. Ges. Wiss. Göttingen math.-phys. Kl. – 1908. – P. 53–111.

[55] Minkowski Hermann // Physikalische Zeitschrift. – 1909. – V. 10. – P. 104.

[56] Mitskievich Nikolai V., Liénard – Wiechert solution revisited. – 2005. arXiv: physics/05...

[57] Mitskievich Nikolai V., Relativistic Physics in Arbitrary Reference Frames. – New York: Nova Publisher. – 2006. gr-qc/9606051.

[58] Mitskievich Nikolai V., The identical physical and geometrical sense of transformations of Galileo and Lorentz // Hadronic Journal – accepted 2007. – P. 1–17.

[59] Mocanu Constantin I., Some difficulties within the framework of relativistic electrodynamics // Archiv für Elektrotechnik, Springer-Verlag – 1986. – V. 69. – P. 97–110.

[60] Mocanu Constantin I., On the relativistic velocity composition paradox and the Thomas rotation // Foundations of Physics Letters – 1992. – V. 5, № 5. – P. 443–456.

[61] Möller C., The Theory of Relativity. – Ofxord: Clarendon Press. – 1960, 1972.

[62] Oziewicz Zbigniew, Classical field theory and analogy between Newton's and Maxwell's equations // Foundation of Physics – 1994. – V. 24, № 10. – P. 1379–1402. hep-th/9312009.

[63] Oziewicz Zbigniew, Boost-link-problem: the complete solution // XLVIII Congreso Nacional de Sociedad Mexicana de Física, Universidad de Guadalajara. Suplemento del Boletino, Abstracts. – 2005. – V. 19. – № 3. – P. 136.

[64] Oziewicz Zbigniew, How do you add relative velocities? In: Pogosyan George S., Luis Edgar Vicent and Kurt Bernardo Wolf, Editors // Group Theoretical Methods in Physics, Institute of Physics. – 2005. Conference Series Number 185, Bristol 2005.

[65] Oziewicz Zbigniew, Isometry boost-link problem: the complete solution, Foundations of Physics – accepted 2006. – P. 1–34.

[66] Oziewicz Zbigniew, The Lorentz boost-link is not unique, math-ph/0608062.

[67] Oziewicz Zbigniew, What is categorical kinematics? International Journal of Geometric Methods in Modern Physics – 2007. – V. 4, № 1 – P. 1–10. math.CT/0608770.

[68] Page William // http://wiki.axiom-developer.-org/SandBoxCategoricalRelativity. – 2006. – P. 1–12.

[69] Panofsky W. K. H., and M. Philips, Classical Electricity and Magnetism. – Mass. Addison-Wesley, Reading. – 1962.

[70] Pervushin Victor N., Creation of the universe // physics/0512273 version 2. – 11 March 2006.

[71] Plebański Jerzy F., and Maciej Przanowski, Notes on a cross product of vectors // Journal of Mathematical Physics. – 1988. – V. 29, № 11. – P. 2334–2337.

[72] Poincaré Henri, Electricity and Optics. – Paris. – 1890, 1901.

[73] Ramakrishnan Alladi, Einstein — a natural completion of Newton // Journal of Mathematical Analysis and Applications – 1973. – V. 42, № 2. – P. 377–380.

[74] Ramakrishnan Alladi, Unnoticed symmetries in Einstein's special relativity // Journal of Mathematical Analysis and Applications – 1978. – V. 63, № 2. – P. 335–338.

[75] Ramakrishnan Alladi, A new concept in special relativity: exterior relative velocity // Proceedings of Tamil Nadu Academy of Sciences – 1978. – V. 1. – P. 67.

[76] Ramakrishnan Alladi, Further unnoticed symmetries in special relativity // Journal of Mathematical Analysis and Applications – 1983. – V. 94, № 1. – P. 237–241.

[77] Riemann Bernhard // Nature – 1873 (May 1). – P. 14–18, 36–37.

[78] Robertson H. P. // Review of Modern Physics – 1949. – V. 21. – P. 378.

[79] Rodrigues Waldyr Alves Jr., and Edmundo Capelas de Oliviera, The many faces of Maxwell, Dirac and Einstein equations, A Clifford bundle approach. Draft – 2005. http://www.ime.unicamp.br/rel_pesq/-2005/rp56-05.html.

[80] Schrödinger Ernst, Specielle Relativitätstheorie und Quantenmechanik. // Sitzungsber. Preuss. Akad. Wiss. phys.-math. Kl. – 1931. – Bd. – V. 12. – P. 238–247.

[81] Sommerfeld Arnold, Electrodynamics. Lectures on Theoretical Physics, Volume III. – Leipzig: Akademie Verlag. – 1945. – New York: Academic Press. – 1952.

[82] Synge John Lighton, Relativity: The General Theory. – Amsterdam: North-Holland Publishing Company. – 1960.

[83] Świerk Dariusz, Relativity theory and product structures. // Master Thesis, Uniwersytet Wrocławski, Instytut Fizyki Teoretycznej. – 1988. – P. 1–42.

[84] Tolman R. C., Relativity, Thermodynamics and Cosmology. – Oxford: Clarendon. – 1918. – 1934.

[85] Ungar Abraham A., The Thomas rotation formalism underlying a nonassociative group structure for relativistic velocities // Applied Mathematics Letters. – 1988. – V. 1, № 4. – P. 403–405.

[86] Ungar Abraham A., Successive Lorentz transformations of the electromagnetic field // Foundations of Physics. – 1991. – V. 21, № 5. – P. 569–589.

[87] Ungar Abraham A., Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces. – Boston: Kluwer Academic. – 2001.

[88] Vargas José Gabriel, Comment on Lorentz transformations from the first postulate // American Journal of Physics – 1976. – V. 44, № 10. – P. 999–1000.

[89] Vargas José Gabriel, Comments on Mansouri and Sexl's test theory of special relativity //Revista Colombiana de Física – 1980. – V. 14, № 1. – P. 79–89.

[90] Vargas José Gabriel, Special relativity, the rod contaction, clock retardation, ether theory, and the conventionality of synchronization // Revista Colombiana de Física – 1980. – V. 14, № 1. – P. 91–112.

[91] Vargas José Gabriel, Nonrelativistic para-Lorentzian mechanics // Foundations of Physics – 1981. – V. 11, №. (3/4). – P. 235–278.

[92] Vargas José Gabriel, Spontaneous para-Lorentzian conserved-vector and nonconserved-axial weak currents // Foundation of Physics – 1982. – V. 12, № 8. – P. 765–779.

[93] Vargas José Gabriel, Nonrelativistic para-Maxwellian electrodynamics with preferred reference frame in the universe // Foundation of Physics – 1982. – V. 12, № 8. – P. 889–905.

[94] Vargas José Gabriel, Revised Robertson's test theory of special relativity // Foundation of Physics – 1984. – V. 14, №. 7. – P. 625–651.

[95] Vargas José Gabriel, Kinematical and gravitational analysis of the Rocket-Borne clock experiment by Vessot and Levine using the revised Robertson's test theory of special relativity // Foundation of Physics – 1986. – V. 16, № 10. – P. 1003–1020.

[96] Vargas José Gabriel, and Douglas G. Torr, Revised Robertson's test theory of special relativity: space-time structure and dynamics // Foundation of Physics – 1986. – V. 16, № 11. – P. 1089–1126.

[97] Vargas José Gabriel, Revised Robertson's test theory of special relativity: supergroups and superspace // Foundation of Physics – 1986. – V. 16, № 12. – P. 1231–1261.

[98] Vargas José Gabriel, and Douglas G. Torr, Testing of the line element of special relativity with rotating systems // Physical Review – 1989. – V. A39, № 6. – P. 2878–2887.

[99] Vargas José Gabriel, and Douglas G. Torr, Syncronizations versus simultaneity relations, with implications for interpretations of quantum measurements. – 2000.

[100] Voigt W., Über das Doppler'sche Princip // Nachrichten Ges. Wiss. Göttingen. – 1887. – P. 41.

[101] Weitzenböck Roland, Invariantentheorie. – Gröningen: – 1923.

[102] Wesley J. P., Michelson-Morley result, a Voigt-Doppler effect in absolute space and in absolute time // Foundations of Physics – 1986. – V. 16. – P. 817–824.

[103] Zel'manov A. L., Monad formalism, Doklady Akademii Nauk SSSR. – 1976. – V. 227. – P. 78.