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$O\,(2)$-equivariant Covering Spaces

In memory of Juan José Rivaud Morayta (1943–2005)

Contents

Abstract

This paper has expository nature and it is a survey on the meaning of symmetry for a topologist. The main purpouse is to give an approximation to the precise concept of equivariant fundamental grupoid starting from the one of ordinary fundamental grupoid. This concept is closely related to the one of equivariant covering space, so we start also dealing with ordinary covering spaces. For didactical reasons some classical results are presented, as well as some relatively new. A few proof are inserted when they are short enough and contribute with clarity.

There is a very strong interaction between the concepts of fundamental group and covering space for connected topological spaces, and something similar happens between fundamental grupoid and covering space for the non connected ones. In the equivariant landscape, fundamental grupoid keeps having this relationship, but now it has to be formulated in terms of the language of categories. Let us recall that a *small category* is one on which the objects constitute a set, and that a *grupoid* is a category where all the morphisms are isomorphisms. A subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is said to be a *full subcategory* if given two objects in $\mathcal{A}$, the morphisms between them in $\mathcal{A}$ are all the possible morphisms between them in $\mathcal{B}$.

An *isomorphism* of categories is a functor which is a bijection on both objects and morphisms. Connected with the notion of isomorphism we find the notion of equivalence of categories but before we need some previous definitions. A functor $T : \mathcal{A} \to \mathcal{B}$ is said to be a *full functor* if for every pair of objects $a_1, a_2 \in Ob\,(\mathcal{A})$ and every morphism $\beta : T\,(a_1) \to T\,(a_2)$ there is a morphism $\alpha : a_1 \to a_2$ such that $\beta = T\,(\alpha)$. A functor $T : \mathcal{A} \to \mathcal{B}$ is a *faithful functor* if for every pair of objects $a_1, a_2 \in Ob\,(\mathcal{A})$ and every pair of morphisms $\alpha_1, \alpha_2 : a_1 \to a_2$ happens that $T\,(\alpha_1) = T\,(\alpha_2) : T\,(a_1) \to T\,(a_2)$ implies $\alpha_1 = \alpha_2$. A functor $T : \mathcal{A} \to \mathcal{B}$ is said to be an *equivalence of categories* if it is full, faithfull and each object $b \in Ob\,(\mathcal{B})$ is isomorphic to an object $T\,(a)$ for some object $a \in Ob\,(\mathcal{A})$.

Another useful tool is the *skeleton* [9] of a category: given a category $\mathcal{C}$, a skeleton $sk\,\mathcal{C}$ is a full subcategory with exactly one object from each isomorphism class in $\mathcal{C}$. The inclusion functor $I : sk\,\mathcal{C} \to \mathcal{C}$ is not an isomorphism, but it is an equivalence of categories: it is clearly full and faithful since the only morphisms in $sk\,\mathcal{C}$ are the identity ones $1 : c \to c$, on the other hand, every object is isomorphic to the representant of its isomophism class in the skeleton. The obvious projection functor $P : \mathcal{C} \to sk\,\mathcal{C}$ is an inverse functor, and . The composite $P \circ I$ is clearly the identity functor.

There is a category whose objects are topological spaces and whose morphisms are continuous maps. Any subcategory of topological spaces and continuous maps is called a topological category. For a complete exposition of these concepts the reader may want to refer to the classical text by Saunders Mac Lane [6] or the excelent book by Peter may [9].

1. Introduction

Let X be a metric space, so it is a Hausdorff topological space. It is said that two continuous maps $f_0, f_1 : X \to X$ are *homotopic maps* which is written by $f_0 \simeq f_1$ if there exists a continuous map $F : X \times I \to X$ such that $F\,(x, 0) = f_0\,(x)$ and $F\,(x, 1) = f_1\,(x)$. The map F is said to be a homotopy between f_0 and f_1. Let us recall that $\pi_0\,(X)$ is the set of *homotopy classes* of pointed maps $\alpha : S^0 \to X$, so one may think about $\pi_0\,(X)$ as the set of path-connected components of the topological space X. It is very well known that $\pi_0\,(X)$ has not in general the structure of a group, unless X is an H-space. The best known example is the *fundamental group*, since $\pi_1\,(X) = \pi_0\,(\Omega\,(X))$ where $\Omega\,(X)$ is the H-space of the based loops on the space X.

An *H-space* also called *H-group* is a non vacuous set G together with an operation $\mu : G \times G \to$ satisfying the group axioms up to homotopy. That is, for example, in a group the associativity means that

$$(\mu \times 1) \circ \mu = (1 \times \mu) \circ \mu$$

where 1 stands for the identity map, or equivalently, the followig diagram commutes

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{1 \times \mu} & G \times G \\ {\scriptstyle \mu \times 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G \end{array}.$$

In an H-group the above diagram commutes up to homotopy, that is:

$$(\mu \times 1) \circ \mu \cong (1 \times \mu) \circ \mu$$

We say that an H-group is H-associative. Analogously, an H-group has H-identity and H-inverses. These concepts may be consulted in any text of Algebraic Topology as for instance the excelent one by Rotman [11].

If G is a compact Lie group and $H < G$ is the identity component then $H \triangleleft G$ so the quotient G/H is a group and this provides $\pi_0\,(G)$ with a group structure, since

$$\pi_0\,(G) = G/H.$$

A particularly interesting case is the one of the orthogonal groups, since they have exactly two path-connected components, so the group of path-connected components is the cyclic of order two, that is

$$\pi_0\,(O\,(n)) \cong C_2.$$

This can be also seen as a short exact sequence

$$1 \to SO\,(n) \longrightarrow O\,(n) \longrightarrow C_2 \to 1$$

for $n \geq 2$. For $n = 2$ we have that $SO\,(2)$ is the circle group, so $O\,(2)$ may be seen as a sort of continuous dihedral gruop.

It is a very well known fact that for finite dimensional Banach spaces every *isometry* A with $A\,(0) = 0$ is a lineal transformation, so in particular, every isometry $A : \mathbb{R}^n \to \mathbb{R}^n$ is such that $A \in O\,(n)$. Thus we have that the isometry group of $\mathbb{R}^n$ is precisely $O\,(n)$. Therefore, if $A \in O\,(n)$, then for every $x \in \mathbb{R}^n$ it happens that $\langle Ax, Ax \rangle = |x|^2$ and also $\langle Ax, Ax \rangle = \langle x, A^t Ax \rangle$ so there is a constant $k \in \mathbb{R}$ such that $A^t A = kI$, but since $\det A^t = \det A$ then $k = 1$ and $A^t A = I$.

Let us denote by r the reflection over the first coordinate

$$r\,(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$$

then the two path-connected components of $O\,(n)$ may be written as the union $[1] \cup [1]$ where $[r]$ is the identity component $SO\,(n)$ and $[r]$ which is the coset $rSO\,(n)$.

Let X be a topological subspace of $\mathbb{R}^n$, a *symmetry* of X is an isometry $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A\,(X) = X$, and it is clear that the symetries of X built a group with the map composition as group operation. Therefore, it follows that the *symmetry group* of X is a subgroup of the orthogonal group $O\,(n)$.

Let us note that $\mathcal{M}\,(X)$ the set of continuous maps from X to X, the homotopy is an equivalence relation in $\mathcal{M}\,(X)$, where the set of homotopy classes is denoted by $[X, X]$ so

$$\frac{\mathcal{M}\,(X)}{\simeq} = [X, X]\,.$$

Given a homotopy $F : X \times I \to X$, for a fixed $t \in I$ we have that $f_t \in \mathcal{M}\,(X)$ where $f_t\,(x) = F\,(x, t)$, thus the homotopy F infuces a map $\hat{F} : I \to \mathcal{M}\,(X)$, which is continuous if $\mathcal{M}\,(X)$ is given the compact-open topology. This means that $\hat{F}$ is a path in $\mathcal{M}\,(X)$, so the elements of $[X, X]$ are the path components of $\mathcal{M}\,(X)$, that is

$$[X, X] = \pi_0\,(\mathcal{M}\,(X))\,.$$

Theorem 1. *The euclidean space $\mathbb{R}^n$ has exactly two homotopy classes of isometries.*

Proof. It is clear since $\pi_0\,(O\,(n)) = C_2$. ■

For details, let us pay attention to the following diagram. The composition $\det \circ \alpha$ is continuous if and only if α is. For α continuous, since the interval $I = [0,1]$ is path-connected, this composition is a constant map.

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & O(n) \\ & \searrow^{\det \circ \alpha} & \downarrow \det \\ & & \mathbb{Z} \end{array}$$

Each possible value of $\det \circ \alpha$ corresponds to a path-connected component. A very good introduction to Homotopy theory may be found in Fulton [3] and for a classical and neat exposition on classical groups the reader may want to go to tom Dieck [12].

2. The Ordinary Fundamental Grupoid

A standard way of making out a monoid $\mathcal{M}$ from a set X with an equivalence relation $\sim$ on it, is taking as objects the points of X and considering that for $x, y \in X$ there is a morphism $x \to y$ if and only if $x \sim y$. Therefore, for any topological space X we have its *fundamental grupoid* $\Pi(X)$ is the small category where $Ob\Pi(X) = X$ and $Mor\Pi(X)$ is the set of arrows $x \to y$ where x and y are the endpoints of a path in X, that is, there exists a path $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$.

The point here is that the morphisms are concentrated in path-connected components, that is, a morphism exists only between two objects in the same path-connected component, and since any path is reversible, then any morphism is an isomorphism, so $\Pi(X)$ is indeed a grupoid. Furthermore, according to the terminology of categories, it is clear that $\pi_0(X) = sk\Pi(X)$, and this is possibly the reason to call sometimes fundamental grupoid to its skeleton.

A *fibration* with *fiber* F is a continuous map $p : E \to X$ called *projection* such that for each $x \in X$ there exist an *admissible neighbourhood* $U \in \mathfrak{U}_x$ and a homeomorphism $\varphi_U : p^{-1}(U) \to U \times F$ which is also a homeomorphism restricted to each fiber, that is, $\varphi_U|_x : p^{-1}(x) \to \{x\} \times F$. Here the symbol $U \in \mathfrak{U}_x$ stands for the set of all the neigbourhoods of the point x, that is the collection of all the open sets containing x.

The space E is called *total space* and the space X is called *base space*. If the total space is homeomorphic to the product $X \times F$ the fibration is said to be trivial so any fibration is *locally trivial*, that is, it is trivial on any admissible neigbourhood. It is usual to denote the fibration (E, p, X) by a vertical diagram as follows where the map ι is the inclusion:

$$\begin{array}{c} F \\ \iota \downarrow \\ E \\ p \downarrow \\ X \end{array}$$

A *covering projection* is a fibration with discrete fibre. In this case the total space is called *covering space*, and if the covering space is simply connected it is said to be a *universal cover* for the space X and it is denoted by $\tilde{X}$. For this case the notation for the covering space $\left(\tilde{X}, p, X\right)$ takes the folowing form:

$$\begin{array}{c} F \\ \iota \downarrow \\ \tilde{X} \\ p \downarrow \\ X \end{array}$$

Example 1. *1. The exponential map $exp : \mathbb{R} \to S^1$ is a covering projection with fiber $\mathbb{Z}$.*

2. The exponential map $exp \times exp : \mathbb{R}^2 \to S^1 \times S^1$ is a covering projection with fiber $\mathbb{Z} \times \mathbb{Z}$.

3. The Hopf map $h : SU(2) \to SO(2)$ is a covering projection with fiber C_2.

4. In general, the projection $p : S^n \to \mathbb{R}P^n$ tal que $p^{-1}([x]) = \pm x$ is a covering projection with fiber C_2.

5. The projection $p : S^{2n+1} \to \mathbb{C}P^n$ is a fiber bundle and has fiber S^1, so it is not a covering projection.

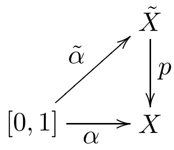
6. The projection $p : M \to S^1$ where M is the Möbius strip is a fibration but not a covering projection, since its fiber is $\mathbb{R}$.

7. Given a topological group G acting on a topological space X, the quotient projection $q : X \to X/G$ is a fibration which is a covering projection just when G is discrete.

The following propositions are very well known results and may be consulted in [3, 11] or [9].

Proposition 2. *In a covering space with connected base space all the fibers have the same cardinality.*

Proposition 3. *Given a loop $\alpha : \left(S^1, 1\right) \to (X, x_0)$, and a covering projection $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$, there exists a unique lifting $\tilde{\alpha} : [0,1] \to \tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}_0$.*



A *lifting* of the map α is a map $\tilde{\alpha}$ making the above diagram to commute.

Proposition 4. *In an universal covering, two liftings α and β with $\alpha(0) = \beta(0)$ are homotopic if and only if they have the same endpoint, that is $\alpha(1) = \beta(1)$.*

From the above result it is quite simple to deduce the following property which is very helpful in the calculation of homotopy groups.

$$|\pi_1(X)| = \#\left(p^{-1}(x)\right).$$

Let us go now to the equivariant scenario.

3. The Orbit Category

Let G be a compact Lie group, in which follows, all its subgroups are understood to be closed. An *action* of G on a topological space X is a group homomorphism $\mu : G \to Aut(X)$ where $Aut(X)$ is the group of self-homeomorphisms of X. Once a G-action is defined on X we say that X is a *G-space*. It is used to white gx instead of the more cumbersome $\mu(g)(x)$. Given two G-spaces X and Y, an *equivariant map* or simply a *G-map* $f : X \to Y$ is one such that commutes with the G-action, that is, such that $f(gx) = gf(x)$ for every $(g, x) \in G \times X$.

The *orbit* of $x \in X$ is defined as the set $G(x) = \{gx | g \in G\}$. The *isotropy group* of an element $x \in X$ is $G_x = \{g \in G | gx = x\}$ and it is quite easy to see that the quotient space of left cosets G/G_x is equivariantly homeomorphic to $G(x)$. For a subgroup $H < G$ the *fixed point set* under H is the set $X^H = \{x \in X | hx = x \forall h \in H\}$. The set of orbits of a given G-action on the topological space X is denoted by X/G and becomes an ordinary topological space under the quotient topology induced by the projection $p : X \to X/G$ given by $p(x) = G(x)$.

An action is *free* it has not fixed points, that is, if $X^G = \varnothing$. Free actions are very important in equivariant algebraic topology, since they are the actions used by the groups to reflect themselves inside a topological space.

The orbit category $\mathcal{O}_G$ is the topological category whose objects are the orbit G-spaces G/H which are in fact compact manifolds, and whose morphisms are G-maps between orbits. Thus we have for $H, K < G$ that

$$Map_G(G/H, G/K) = Mor\mathcal{O}_G.$$

Before going further, we have to make some remarks about the behavior of the orbit spaces, which are the basic transitive G-spaces, and we start with the following little result.

Proposition 5. *Let G be a compact Lie group, for closed subgroups H and K there exists an equivariant map $f : G/H \to G/K$ if and only if H is subconjugate to K.*

Proof. Let f be such an equivariant map, and assume $f(eH) = g_0K$, since f is equivariant, then for all $h \in H$ we have $f(hH) = hf(H) = hg_0K$ which implies $g_0^{-1}hg_0 \in K$. Conversely, if $g_0^{-1}Hg_0 \subseteq K$ for some $g_0 \in G$ then we may define $f : G/H \to G/K$ by $f(gH) = gg_0K$ which is clearly G-equivariant. ∎

Let us consider two closed subgroups H ahd K; there is an obvious action $H \times G/K \to G/K$ given by left multiplication $(h, gK) \mapsto hgK$. If $gK \in (G/K)^H$ then $hgK = gK$ for all $h \in H$ so $(H) < (K)$ and the converse is obvious. This associates uniquely an equivariant map by using the above result with a fixed point of this action so we have

$$Map_G(G/H, G/K) = Mor\mathcal{O}_G = (G/K)^H$$

which shows an intimate relationship between equivariant maps and fixed point sets, in particular, if H is not a subconjugate to K then G/K is a free H-space.

A G-space X is said to be *transitive* if it has only one orbit, that is, if $G(x) = X$ for any $x \in X$. From the above result it becomes clear that there is a G-homeomorphism of transitive G-spaces if and only if $(H) = (K)$, it is to say, if the subgroups H and K belong to the same conjugacy class.

We may construct the category of covering spaces over transitive G-spaces. An *equivariant covering space* is a G-fibration with discrete fibres. The fibre is assumed to be either a discrete space or a *homotopy discrete space*, this means, a space such that the discretisation map $F^K \to \pi_0\left(F^K\right)$ is continuous for every closed subgroup K, and a homotopty equivalence. For the second case we call the fibrartion a *homotopy G-covering space.*

Example 2. *Let $G = C_2$ be the cyclic group of order 2. There is an action of G on the real line $\mathbb{R}$ where the generator $r \in C_2$ acts on any $x \in \mathbb{R}$ by $rx = -x$. On the other hand, there is an action of G on the circle $S^1 \subseteq \mathbb{C}$ by complex conjugation, that is $rz = \overline{z}$. The ordinary exponential projection* $\exp : \mathbb{R} \to S^1$ *given by* $\exp(x) = e^{ix}$ *is a C_2-equivariant covering projection.*

Example 3. *The center of $SU(2)$ is C_2, and this provides a C_2-action on $SU(2)$. By regarding $SO(3)$ as a trivial C_2-space, the Hopf projection is a C_2-equivariant covering map.*

Let F be a homotopy discrete H-space, thus F is an H-space, so is G, therefore, the quotient

$$E = G \times_H F = (G \times F)/H$$

is a covering space over G/H, and it has fiber $F_0 = F/H$.

In order to keep control, Costenoble and Waner [2] restric their study to a category of equivariant covering spaces that they write as $\mathcal{D}_G$. Let G be a compact Lie group, the objects in the category $\mathcal{D}_G$ of covering spaces over G-orbits have the form $G \times_H F$ and the corrsponding morphisms are fibre preserving equivariant maps.

In the non-equivariant setting, the existence of universal covering spaces, as we have seen before is crucial to recover the homotopy of a given topological space. This is the main reason why one would expect that whenever one has a universal cover $p : E \to X$ the restriction to fixed point sets $p^H : E^H \to X^H$ would be again universal covers, but this is not the case as one can see through the following two examples.

Example 4. *Let $G = C_2$ acting on $X = S^2$ by $r(x, y, z) = (x, y, -z)$ thus $X^G = S^1$. The identity map $p : S^2 \to S^2$ is an universal cover but clearly $X^G = S^1$ and the restriction $p^G : S^1 \to S^1$ is also the identity but it is not a universal cover.*

Example 5. *Let $G = C_2$ acting on $X = S^1$ by $r(x, y) = (x, -y)$ thus $X^G = \{+1, -1\}$. If the real line is given the natural action by $G = C_2$. The exponential map $exp : \mathbb{R} \to S^1$ is an equivariant universal covering space. The map $p^G : \{0\} \to X^G$ is an universal cover over $\{+1\}$ but it is not over $\{-1\}$ since $0 \notin p^{-1}(-1)$.*

4. 1-dimensional groups

Let us take first the case of the circle group $G = \mathbb{T}$, if E is a connected covering space over the circle group, then the projection $p : E \to G$ induces an injective homomorphism $p_* : \pi_1(E) \to \pi_1(G)$, thus E has to be either the real line or else a circle and so that the projection map is either the exponential or else an integer power.

Let us deal with the non universal coverings. For this case it is clear that the non unversal covering spaces over the circle group have the form $E = \mathbb{T} \times_{C_n} C_n = \mathbb{T}$ whenever the projection map is $z \mapsto z^n$. The universal covering space for the circle may not be recovered as one of these bundles since it wiold have the form $\mathbb{T} \times_1 \mathbb{Z}$ which is not even connected. From this one can see that the classification given by Costenoble and Waner [2] need to be completed.

Now since the circle group is abelian there is a conjugacy class for each closed subgroup, so the diagram of possible morphisms in the category $\mathcal{O}_{\mathbb{T}}$ is the following.

$$\mathbb{T}/1 \circlearrowleft \longrightarrow \mathbb{T}/C_k \circlearrowleft \longrightarrow \mathbb{T}/C_{nk} \circlearrowleft \longrightarrow \mathbb{T}/\mathbb{T} \circlearrowleft$$

In fact, as $\mathbb{T}$-sets one has

$$Map_{\mathbb{T}}(\mathbb{T}/C_n, \mathbb{T}/C_{nk}) \cong (\mathbb{T}/C_{nk})^{C_n} \cong C_k/1$$

for $k > 1$ and for $k = 1$ it is clear that $\mathbb{T}/C_n$ is C_n-fixed so

$$Map_{\mathbb{T}}(\mathbb{T}/C_n, \mathbb{T}/C_n) \cong (\mathbb{T}/C_n)^{C_n} \cong \mathbb{T}/C_n.$$

It is well known that the conjugacy classes for $G = O(2)$ are represented by the cyclic and the dihedral groups, then the the arrows in the following diagram show all the possible morphisms in the orbit category $\mathcal{O}_{O(2)}$

$$\begin{array}{ccccccc}
G/1 \circlearrowleft & \longrightarrow & G/C_k \circlearrowleft & \longrightarrow & G/C_{nk} \circlearrowleft & \longrightarrow & G/\mathbb{T} \circlearrowleft \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G/D_2 \circlearrowleft & \longrightarrow & G/D_{2k} \circlearrowleft & \longrightarrow & G/D_{2nk} \circlearrowleft & \longrightarrow & G/G \circlearrowleft
\end{array}$$

The following diagram shows all the possible morphisms in the orbit category $\mathcal{O}_Q$.

$$\begin{array}{ccccccc}
Q/1 \circlearrowleft & \longrightarrow & Q/C_{m-1} \circlearrowleft & \longrightarrow & Q/C_m \circlearrowleft & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Q/C_2 \circlearrowleft & \longrightarrow & Q/C_{2m-2} \circlearrowleft & \longrightarrow & Q/C_{2m} \circlearrowleft & \longrightarrow & G/\mathbb{T} \circlearrowleft \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q/Q_4 \circlearrowleft & \longrightarrow & Q/Q_{4m-4} \circlearrowleft & \longrightarrow & Q/Q_{4m} \circlearrowleft & \longrightarrow & Q/Q \circlearrowleft
\end{array}$$

As it has been shown in [4], to deal with the 1-dimensional groups it is enough to deal with the three groups considered in this section.

5. The Equivariant Fundamental Grupoid

Let X be a G-space. The *equivariant fundamental grupoid* is the category $\Pi_G(X)$ whose objects are equivariant maps from the canonical transitive G-spaces to X, this fact can be written by $Ob\Pi_G(X) = \{x : G/H \to X\}$. The idea behind this definition is that for each $x \in X^H$ with orbit $G(x) \subseteq X^H$ it is defined an equivariant map $x : G/H \to G(x)$ such that $x(1H) = x$ where in the last equation we use both meanings of the symbol x. To carry on we need to precise the meaning of the symbol X^H, and we define it as the set of points $z \in X$ such that the isotropy group G_z belongs to the conjugacy class (H).

The morphisms of the category $\Pi_G(X)$ are pairs $([\omega], \alpha) : x \to y$ where $\alpha : G/H \to G/K$ is an equivariant map for $x : G/H \to X$ and $y : G/K \to X$, and $[\omega]$ is the homotopy class of a the path $\omega : I \to X^H$ with $\omega(0) = x(1H)$ and $\omega(1) = y(1K)$. We denote $Mor\Pi_G(X) = \{([\omega], \alpha) : x \to y\}$.

This definition also needs an explanation. Note first that for each equivariant map $\alpha : G/H \to G/K$ there is one and only one equivariant map $\alpha_0 : G\,(1H) \to G\,(1K)$ making the following diagram to commute.

$$\begin{array}{ccc} G/H & \xrightarrow{x} & G\,(x) \\ \alpha\downarrow & & \downarrow\alpha_0 \\ G/K & \xrightarrow[y]{} & G\,(y) \end{array}$$

Note that α exists if and only if H is subconjugate to K and in that case $X^K \subseteq X^H$ so ω mantains itsef inside X^H. On the other hand, there is a projection functor $\pi : \Pi_G\,(X) \to \mathcal{O}_G$ given by $\pi : (x : G/H \to X) = G/H$ and $\pi\,([\omega]\,,\alpha) = \alpha$ for which $\pi^{-1}\,(G/H) = \Pi\left(X^H\right)$, that is the ordinary fundamental grupoid, so the fundamental grupoid is indeed a sort of *bundle of grupoids* over $\mathcal{O}_G$. The reader would note that it is an actual bundle in the ordinary sense only in the case that G is the trivial group. Nevertheless, in the equivariant setting a category as the fundamental grupoid with no fixed fibre will be calles a bundle of grupoids.

$$\begin{array}{ccc} x & \xrightarrow{\pi} & G/H \\ ([\omega]\,,\alpha)\downarrow & & \downarrow\alpha \\ y & \xrightarrow[\pi]{} & G/K \end{array}$$

In order to clarify these concepts note the commutativity of the following diagram. The concept of equivariant fundamental grupoid was first used by Grothndieck in [5] and May have produced further developments in [7] and [8].

6. Discrete Representations

The analogue of the fibre bundles with discrete fibre for equivariant topological spaces is the concept of discrete representation. Let $\mathcal{D}_G$ be the category whose objects are the G-covering spaces over the orbits of G, that is, over the transitive G-spaces, and whose morphisms are the G-fiber homotopy equivalences, that is, the objects in $\mathcal{D}_G$ have the form $G \times_H F$, so there is a projection map $G \times_H F \to G/H$ defining a functor $p : \mathcal{D}_G \to \mathcal{O}_G$. A *discrete representation* is a continuous covariant functor over $\mathcal{O}_G$, that is, a functor $F : \Pi_G\,(X) \longrightarrow \mathcal{D}_G$ such that $F \circ p = \pi$.

$$\begin{array}{ccc} \Pi_G\,(X) & \xrightarrow{F} & \mathcal{D}_G \\ & \searrow^{\pi} & \downarrow p \\ & & \mathcal{O}_G \end{array}$$

We shall denote by $\mathcal{R}\,(X)$ the category of *discrete representations* of $\Pi_G\,(X)$ and natural isomorphisms and let us denote by $\mathcal{C}\,(X)$ the category of *G-covering spaces* and fibrewise G-homeomorphisms.

The following analogue of proposition 4 has been proven by Costenoble and Waner in [2] which is valid for compact connected Lie groups.

Theorem 6. *Given a G-space X of the G-homotopy type of a G-CW complex, there are functors*

$$\Delta : \mathcal{C}\,(X) \longrightarrow \mathcal{R}\,(X)$$

and

$$\nabla : \mathcal{R}\,(X) \longrightarrow \mathcal{C}\,(X)$$

that are inverse equivalences of categories.

This result tells us that there is a bijection between the equivalence classes of G-covering spaces and the isomorphism classes of discrete representation of $\Pi_G\,(X)$. There is a thirth equivalent category as one can see in the next result appearing also in [2]. A discrete $\Pi_G\,(X)$*-action* A is a continuous covariant functor $A : \Pi_G\,(X) \longrightarrow$ **Set**. We shall denote by $\mathcal{A}\,(X)$ the category of $\Pi_G\,(X)$-actions and natural isomorphisms.

Theorem 7. *Given a G-space X of the G-homotopy type of a G-CW complex, there are functors*

$$\Phi : \mathcal{C}\,(X) \longrightarrow \mathcal{A}\,(X)$$

and

$$\Psi : \mathcal{A}\,(X) \longrightarrow \mathcal{C}\,(X)$$

that are inverse equivalences of categories.

The above results are valid for G a compact and connected Lie group end their proofs rest on results that appear in Costenonble and Waner (2001) [1] covering only the case when G is a discrete group. For non-connected groups we have to use the fact that a G-CW complex has the equivariant homotopy type of an ordinary CW-complex clear that a G, which has been proven in [10] only for $G = O\,(2)$ and $Q = N_{SU(2)}\,(\mathbb{T})$ the continuous quaternion group, which can be seen as the normalizer of the circle group in the isospin group $SU\,(2)$.

Theorem 8. *For $G = O\,(2)$ or $G = Q$ there are equivalences of categories*

$$\mathcal{R}\,(X) \longleftarrow \mathcal{C}\,(X) \longrightarrow \mathcal{A}\,(X)$$

The machinery developed by Contenoble an Waner works well with connected groups and with bundles of the form $G \times_H F$ but the theory needs to be extended for non-connected groups and more general bundles. This will be published elsewhere.

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