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Charge Conjugation in the Galilean Limit

Contents

Abstract

Strictly working in the framework of the nonrelativistic quantum mechanics of a spin 1/2 particle coupled to an external electromagnetic field, we show, by explicit construction, the existence of a charge conjugation operator matrix which defines the corresponding antiparticle wave function and leads to the galilean and gauge invariant Schroedinger-Pauli equation satisfied by it.

1. Introduction

In a recent paper [1], Cabo *et al* showed the *existence* of the nonrelativistic limit C_{nr} of the charge conjugation operation C for the Dirac equation of a 4-spinor $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^t$ coupled to an external electromagnetic potential $(\phi, \vec{A})$. At low velocities of the Dirac particle with respect to the velocity of light in vacuum c, the "large components" $\psi = (\psi_1, \psi_2)^t$ of Ψ satisfy the Schroedinger-Pauli equation [2]

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix} = \frac{1}{2m}\Bigg(-\nabla^2 + \frac{q^2}{\hbar^2c^2}\vec{A}^2 + \frac{iq}{\hbar c}\nabla\cdot\vec{A} + \frac{2iq}{\hbar c}\vec{A}\cdot\nabla - \frac{q}{\hbar c}\vec{\sigma}\cdot\vec{B} + 2mq\phi\Bigg)\begin{pmatrix}u\\v\end{pmatrix} \quad (1)$$

where q and m are the electric charge and mass respectively, $\vec{\sigma}$ are the Pauli matrices, $\vec{B} = \nabla \times \vec{A}$ is the magnetic field and, at each space time point $(\vec{x}, t)^t$, $(u, v)^t \in \mathbb{C}^2$. The *charge conjugate* Pauli spinor ψ_c representing spin 1/2 antiparticles (e.g. positrons) if ψ represents spin 1/2 particles (e.g. electrons) is given by

$$\psi_c = \begin{pmatrix}-\bar{v}\\ \bar{u}\end{pmatrix} = C_{nr}\begin{pmatrix}u\\v\end{pmatrix} \quad (2)$$

where

$$C_{nr} = KM, \quad M = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad (3)$$

is the nonrelativistic limit of the charge conjugation matrix of the Dirac equation, which up to a sign is given by [3]

$$C = i\gamma^2\gamma_0 = \begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \quad (4)$$

K is the complex conjugation antilinear and hemitian ($K^\dagger = K$) operation. ψ_c satisfies the equation

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}-\bar{v}\\ \bar{u}\end{pmatrix} = \frac{1}{2m}\Bigg(\nabla^2 - \frac{q^2}{\hbar^2c^2}\vec{A}^2 + \frac{iq}{\hbar c}\nabla\cdot\vec{A} + \frac{2iq}{\hbar c}\vec{A}\cdot\nabla - \frac{q}{\hbar c}\vec{\sigma}\cdot\vec{B} - 2mq\phi\Bigg)\begin{pmatrix}-\bar{v}\\ \bar{u}\end{pmatrix}. \quad (5)$$

As it was proved in reference 1, both (1) and (5) are transformed into each other by the operator C_{nr}, thus reaffirming the *galilean character of the approximation* C_{nr} *to* C. This is a non trivial result specially because of the general belief that charge conjugation is a symmetry that exists only in the relativistic regime. [4]

In this note we discuss the previous result without appealing to the limiting process, namely, strictly working in the context of the galilean group, for simplicity of its connected component G_0, and of its universal covering group $\hat{G}_0$ (section 2). From the lagrangian density $\mathcal{L}$ for the equation (1), and using C_{nr}, we construct the lagrangian density $\mathcal{L}_c$ for equation (5), and prove the galilean invariance of these equations by proving this invariance for $\mathcal{L}$ and $\mathcal{L}_c$. We also verify the gauge invariance of $\mathcal{L}_c$ (section 3).

2. Galilean Group, its Universal Covering Group, and Spinors

The connected component of the galilean group G_0 consists of the set of 4×4 matrices

$$g = \begin{pmatrix} R & \vec{V} \\ 0 & 1 \end{pmatrix} \quad (6)$$

with R in the 3-dimensional rotation group $SO(3)$, boost velocity $\vec{V}$ in $\mathbb{R}^3$, composition law

$$g_2 g_1 = \begin{pmatrix} R_2 & \vec{V}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & \vec{V}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2 R_1 & \vec{V}_2 + R_2 \vec{V}_1 \\ 0 & 1 \end{pmatrix} \quad (6a)$$

identity

$$\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6b)$$

and inverse

$$\begin{pmatrix} R & \vec{V} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}\vec{V} \\ 0 & 1 \end{pmatrix}. \quad (6c)$$

G_0 is a non abelian, non compact, connected but non simply connected six dimensional Lie group; like the connected component of the Lorentz group, its topology is that of the cartesian product of the real projective space with ordinary 3-space *i.e.* of $\mathbb{R}P^3 \times \mathbb{R}^3$. The action of G_0 on spacetime is given by

$$G_0 \times \mathbb{C}^4 \to \mathbb{C}^4,$$
$$\left(g, \begin{pmatrix} \vec{x}' \\ t' \end{pmatrix}\right) \mapsto \begin{pmatrix} \vec{x} \\ t \end{pmatrix} = g \begin{pmatrix} \vec{x}' \\ t' \end{pmatrix} = \begin{pmatrix} R\vec{x}' + \vec{V}t' \\ t' \end{pmatrix}. \quad (7)$$

Since one has the action

$$\mu : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3, \ (R, \vec{x}) \mapsto R\vec{x}, \quad (8)$$

then G_0 is isomorphic to the semidirect sum $\mathbb{R}^3 \times_\mu SO(3)$: $\begin{pmatrix} R & \vec{V} \\ 0 & 1 \end{pmatrix} \mapsto (\vec{V}, R)$ with composition law

$$(\vec{V}', R')(\vec{V}, R) = (\vec{V}' + R'\vec{V}, R'R). \quad (8a)$$

The *universal covering group* of G_0 is given by the $\mathbb{Z}_2$-bundle

$$\mathbb{Z}_2 \to \hat{G}_0 \xrightarrow{\Pi} G_0 \quad (9)$$

where

$$\hat{G}_0 = \{\hat{g} = \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix}, \ T \in SU(2), \ \vec{V} \in \mathbb{R}^3\}, \quad (9a)$$

and Π is the $2 \to 1$ group homomorphism

$$\Pi(\hat{g}) = \begin{pmatrix} \pi(T) & \vec{V} \\ 0 & 1 \end{pmatrix} \quad (9b)$$

with $\pi : SU(2) \to SO(3)$ the well known projection

$$\pi \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, z^2 - \mathrm{Re}\, w^2 & \mathrm{Im}\, z^2 + \mathrm{Im}\, w^2 & -2\mathrm{Re}\, zw \\ -\mathrm{Im}\, z^2 + \mathrm{Im}\, w^2 & \mathrm{Re}\, z^2 + \mathrm{Re}\, w^2 & 2\mathrm{Im}\, zw \\ 2\mathrm{Re}\, z\bar{w} & 2\mathrm{Im}\, z\bar{w} & |z|^2 - |w|^2 \end{pmatrix}. \quad (9c)$$

$\hat{G}_0$ is simply connected and has the topology of $S^3 \times \mathbb{R}^3$. Since $SU(2)$ acts on $\mathbb{R}^3$:

$$\hat{\mu} : SU(2) \times \mathbb{R}^3 \to \mathbb{R}^3, \ (T, \vec{V}) \mapsto \pi(T)\vec{V}, \quad (10)$$

one has the group isomorphism

$$\hat{G}_0 \ni \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix} \mapsto (\vec{V}, T) \in \mathbb{R}^3 \times_{\hat{\mu}} SU(2), \quad (11)$$

the composition law in $\hat{G}_0$ is given by

$$\begin{pmatrix} T' & \vec{V}' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T'T & \vec{V}' + \pi(T')\vec{V} \\ 0 & 1 \end{pmatrix}, \quad (12)$$

while the identity and inverse are respectively given by

$$\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (12a)$$

and

$$\begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} T^{-1} & -\pi(T^{-1})\vec{V} \\ 0 & 1 \end{pmatrix}. \quad (12b)$$

Turning back to physics, for each mass value $m > 0$, $\hat{G}_0$ acts on the infinite dimensional Hilbert space $\mathcal{L}_1^2$ of continuously differentiable and square integrable $\mathbb{C}^2$-valued functions $(u, v)^t$ on $\mathbb{R}^4$, the Schroedinger-Pauli spinors. This action is defined as follows [5]

$$\hat{\mu}_m : \hat{G}_0 \times \mathcal{L}_1^2 \to \mathcal{L}_1^2,$$
$$\left(\begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) \mapsto \begin{pmatrix} T & \vec{V} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} : \mathbb{R}^4 \to \mathbb{C}^2,$$
$$\begin{pmatrix} \vec{x} \\ t \end{pmatrix} \mapsto \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} \vec{x} \\ t \end{pmatrix} = \exp\left(\frac{-im}{\hbar}(\vec{V} \cdot \vec{x} + \frac{1}{2}|\vec{V}|^2 t)\right) \times T \begin{pmatrix} u\left(\pi(T)\vec{x} + \vec{V}t, t\right) \\ v(\pi(T)\vec{x} + \vec{V}t, t) \end{pmatrix}. \quad (13)$$

$\hat{\mu}_m$ is equivalent to the *representation*

$$\tilde{\hat{\mu}}_m : \hat{G}_0 \to End(\mathcal{L}_1^2),$$
$$\tilde{\hat{\mu}}_m(\hat{g}) \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) = \hat{g} \cdot \begin{pmatrix} u \\ v \end{pmatrix}. \quad (13a)$$

At each t one has the inner product

$$\left(\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right)(t) = \int d^3\vec{x}(\bar{u}_2(\vec{x},t)u_1(\vec{x},t) + \bar{v}_2(\vec{x},t)v_1(\vec{x},t)) \quad (14a)$$

and the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 (t) = \left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)(t) = \int d^3\vec{x}(|u(\vec{x},t)|^2 + |v(\vec{x},t)|^2). \quad (14b)$$

The galilean transformation of the charge conjugate spinor ψ_c is given by

$$\psi_c \mapsto \bar{\tilde{g}} \cdot \psi_c, \begin{pmatrix} \bar{T} & \vec{V} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} \begin{pmatrix} \vec{x} \\ t \end{pmatrix} = \exp\left(\frac{im}{\hbar}(\vec{V}\cdot\vec{x} + \frac{1}{2}|\vec{V}|^2 t) \right) \times \bar{T} \begin{pmatrix} -\bar{v}(\pi(\bar{T})\vec{x} + \vec{V}(t,t) \\ \bar{u}(\pi(\bar{T})\vec{x} + \vec{V}(t,t) \end{pmatrix}. \quad (15)$$

Finally, the galilean transformations of the electromagnetic potential $(\phi, \vec{A})$ and the magnetic field $\vec{B}$ are

$$\begin{aligned} \phi(\vec{x},t) &= \phi'(\vec{x}',t'), \\ \vec{A}(\vec{x},t) &= R\vec{A}'(\vec{x}',t'), \\ \vec{B}(\vec{x},t) &= R\vec{B}'(\vec{x}',t') \end{aligned} \quad (16)$$

with $\vec{x} = R\vec{x}' + \vec{V}t'$ and $t = t'$.

Remark: Representations associated with different values of the mass are inequivalent [6].

3. Lagrangian Formulation and Galilean and Gauge Invariances

The Pauli equations (1) and (5) can be formulated within the lagrangian framework. The lagrangian for equation (1) is

$$\begin{aligned} \mathcal{L} = \frac{i\hbar}{2} &\left(\left(\left(\frac{\partial}{\partial t} - \frac{iq}{\hbar}\phi \right) \psi^\dagger \right) \psi - \psi^\dagger \left(\frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi \right) \psi \right) + \frac{\hbar^2}{2m}\left(\nabla + \frac{iq}{\hbar c}\vec{A} \right) \\ &\times \psi^\dagger \cdot \left(\nabla - \frac{iq}{\hbar c}\vec{A} \right) \psi - \frac{q\hbar}{2mc}\psi^\dagger \vec{\sigma}\cdot\vec{B}\psi \\ &= \frac{i\hbar}{2}(\dot{\psi}^\dagger\psi - \psi^\dagger\dot{\psi}) + \frac{\hbar^2}{2m}\nabla\psi^\dagger\cdot\nabla\psi \\ &+ \frac{q^2}{2mc^2}\psi^\dagger|\vec{A}|^2\psi + \frac{i\hbar q}{2mc}\left(\psi^\dagger\vec{A}\cdot\nabla\psi \right. \\ &\left. -\nabla\psi^\dagger\cdot\vec{A}\psi \right) - \frac{q\hbar}{2mc}\psi^\dagger\vec{\sigma}\cdot\vec{B}\psi + q\psi^\dagger\phi\psi, \end{aligned} \quad (17)$$

and equation (1) amounts to the variational equation

$$\frac{\delta}{\delta\psi^\dagger(\vec{x},t)}S = 0 \quad (18)$$

where S is the action

$$S = \int dt \int d^3\vec{x}\mathcal{L}(\vec{x},t). \quad (19)$$

Under the charge conjugation operation

$$\begin{aligned} \mathcal{L} \to \mathcal{L}_c = K\mathcal{L} = -\frac{i\hbar}{2} &\left(\left(\left(\frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi)\psi_c^\dagger \right) \psi_c - \psi_c^\dagger \left(\frac{\partial}{\partial t} - \frac{iq}{\hbar}\phi \right) \psi_c \right) + \frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar c}\vec{A} \right) \\ &\times \psi_c^\dagger \cdot \left(\nabla + \frac{iq}{\hbar c}\vec{A} \right) \psi_c + \frac{q\hbar}{2mc}\psi_c^\dagger\vec{\sigma}\cdot\vec{B}\psi_c \\ &= -\frac{i\hbar}{2}(\dot{\psi}_c^\dagger\psi_c - \psi_c^\dagger\dot{\psi}_c) + \frac{\hbar^2}{2m}\nabla\psi_c^\dagger\cdot\nabla\psi_c \\ &+ \frac{q^2}{2mc^2}\psi_c^\dagger|\vec{A}|^2\psi_c - \frac{i\hbar q}{2mc}\left(\psi_c^\dagger\vec{A}\cdot\nabla\psi_c \right. \\ &\left. -\nabla\psi_c^\dagger\cdot\vec{A}\psi_c \right) + \frac{q\hbar}{2mc}\psi_c^\dagger\vec{\sigma}\cdot\vec{B}\psi_c + q\psi_c^\dagger\phi\psi_c. \end{aligned} \quad (20)$$

To pass from (17) to (20), the identity $M^\dagger M = 1$ is inserted at each term of (17), and the fact that $M\vec{\sigma}M^\dagger = M(\sigma_1,\sigma_2,\sigma_3)M^\dagger = (-\sigma_1,\sigma_2,-\sigma_3)$ is used; then the complex conjugation operation K completes the transformation.

The total action for the particle-antiparticle system is

$$S_{tot} = S + S_c = \int dt \int d^3\vec{x}(\mathcal{L}(\vec{x},t) + \mathcal{L}_c(\vec{x},t)) \quad (21)$$

and equation (5) is obtained from S_{tot} or S_c as

$$\frac{\delta}{\delta\psi_c^\dagger(\vec{x},t)}S_{tot} = \frac{\delta}{\delta\psi_c^\dagger(\vec{x},t)}S_c = 0. \quad (22)$$

The lagrangian $\mathcal{L}$, and therefore the equation (1), are invariant under the galilean transformations (13), (15) and (16) for ψ, ψ_c, and $(\phi,\vec{A})$ and $\vec{B}$, respectively. To prove it, we use the facts that $\nabla = R^{-1}\nabla'$ where $\nabla = \frac{\partial}{\partial\vec{x}}$ and $\nabla' = \frac{\partial}{\partial\vec{x}'}$, and $\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - R^{-1}\vec{V}\cdot\nabla'$. If

$\mathcal{L}'$ and $\mathcal{L}'_c$ are the transformed lagrangian densities for particles and antiparticles, then, from equation (20),

$$\mathcal{L}_c(\vec{x},t) = K\mathcal{L}(\vec{x},t) = K\mathcal{L}'(\vec{x}',t') = \mathcal{L}'_c(\vec{x}',t') \quad (23)$$

and therefore the galilean invariance of equation (5) is also proved.

Finally, both $\mathcal{L}$ and $\mathcal{L}_c$, and therefore the equations (1) and (5), are gauge invariant under the transformations $\psi \to e^{i\Lambda}\psi$, $\psi_c \to e^{-i\Lambda}\psi_c$, $\phi \to \phi - \frac{\hbar}{q}\frac{\partial}{\partial t}\Lambda$ and $\vec{A} \to \vec{A} + \frac{\hbar c}{q}\nabla\Lambda$, where Λ is an arbitrary differentiable function of $(\vec{x},t)$.

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