Theory of Microwaves Amplification and Generation in Coaxial Ubitrons

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Abstract

Results of the theoretical studies addressing microwaves amplification and generation in coaxial Ubitrons with the undulator constructed using permanent circular magnets are presented. Spatial magnetic field structure is designed for a coaxial undulator. High-current relativistic electron beam (REB) transport process in this undulator is considered. Nonlinear theory is constructed for microwaves amplification in the coaxial Ubitron. The efficiency of the Ubitron amplifier is derived. The theory is expanded to consider Ubitron oscillator with an external delayed feedback. Starting currents for this oscillator are determined. Modes of microwave generation are studied for various REB currents. Coaxial backward wave Ubitron oscillator driven by a high-current REB is studied. Initial currents of the oscillator are determined. Nonlinear dynamics of microwave generation for various values of the REB current is considered.

1. Introduction

Basic requirements for modern microwave amplifiers and oscillators, driven by high-current REBs are related to the reduction of the microwave devices weight while preserving their highly efficient output (electron-produced efficiency) and the frequency tuning capability for high output power. Out of the plethora of the microwave devices (carcinotrons, magnetrons, vircators, MILOs and many others of this kind), the most compliant with those requirements are Ubitrons based on permanent magnets. In those Ubitrons the periodic magnetic system performs two functions at once. On the one hand, it is used to focus REB and provide its transport in the electrodynamic system. On the other hand, it is used for high-intensity microwave generation. Very promising in this respect are coaxial Ubitrons, where the microwaves
are amplified when REB passes through the section of the coaxial transmission line [1–4]. In this kind of a line, the REB limiting in-vacuum current increases [5] dramatically as compared to the case of the cylindrical waveguide, which is employed in the conventional Ubitrons. With the dimensions of the drift chamber being the same, this approach allows to achieve greater transport currents and, consequently, greater microwave power output. And, vice versa, at an REB assigned current the overall dimensions of the Ubitron and its weight can be drastically reduced.

The study considers theoretical investigations of the microwave amplification and generation processes in Ubitrons, based on permanent circular magnets. Section 2 adresses spatial structure of a periodic magnetic field of the coaxial undulator. Section 3 presents the results for the theory of high-current REB focusing in the coaxial Ubitron by spatially periodic magnetic field [6]. Section 4 adresses linear and nonlinear steps of microwave amplification in a coaxial Ubitron amplifier [7].

Two experimental options can be implemented for the Ubitron oscillator. The first one involves REB exciting cotraveling electromagnetic wave. The auto-oscillator includes a nonlinear Ubitron amplifier and an external linear (or nonlinear) feedback circuit [8]. According to this design, the signal is applied from the nonlinear amplifier output to its input. Nonlinear dynamics studies of the Ubitron oscillator with external feedback [9] are addressed in Section 5. According to the second design of the Ubitron oscillator, the microwaves generation is produced by backward waves. In this kind of the oscillator, the feedback is distributed. Section 6 deals with the nonlinear dynamics of the Ubitron backward wave oscillator (BWO).

2. Spatial Structure of Coaxial Undulator Periodic Magnetic Field

Coaxial Magnetic Undulator (CMU) represents a structure consisting of two periodically placed circular permanent magnets system with longitudinal magnetization.

The first periodic system is placed inside the central conductor metallic tube (CDC). The second periodic system wraps around the CDC external metallic tube. Circular polar pieces, made of the magnetically soft materials, like iron are placed between the permanent magnets. In this kind of magnetic undulator, the circular magnets changes magnetization vector changes its direction periodically. Internal and external magnetic systems are oriented so that permanent magnets with opposite magnetization face each other. The CMU schematics is shown in Fig. 1. Consider that the thickness of the circular magnets is \( L_n = L_w/4 \), where \( L_w \) is the magnetic system period of CMU. Moreover, the magnetic field strengths of internal and external circular magnets are equal to \( H_0 \). In this event, the CMU magnetic field components are described by the following expressions [6]:

\[
H_z = H_0 \sum_{n=1}^{\infty} \left[ a_n \sin(nq_w\xi) + b_n \cos(nq_w\xi) \right] F_n^{(0)},
\]

\[
H_r = -H_0 \sum_{n=1}^{\infty} \left[ a_n \cos(nq_w\xi) - b_n \sin(nq_w\xi) \right] F_n^{(1)},
\]

where \( a_n = \frac{\cos(\pi n) - 1}{\pi n}, \quad b_n = \frac{2\pi n}{\sin \left( \frac{\pi n}{2} \right)} \),

\( \xi = z/a, \quad k_w = \frac{2\pi}{L_w}, \quad \rho = r/a, \quad F_n^{(0)}(nq_w\rho) = f_n I_0(nq_w\rho) + g_n K_0(nq_w\rho), \quad F_n^{(1)}(nq_w\rho) = f_n I_1(nq_w\rho) + g_n K_1(nq_w\rho), \quad q_w = k_w a, \quad g_n = I_0(nq_w s) + I_0(nq_w), \quad f_n = \frac{\Delta_n}{\Delta_n} = I_0(nq_w s) K_0(nq_w s) - I_0(nq_w s) K_0(nq_w), \)

\( s = b/a, \quad \Delta_n = I_0(nq_w s) K_0(nq_w s) - I_0(nq_w s) K_0(nq_w), \quad I_n(x), \) where \( K_n(x) \) are modified cylindrical functions.

Fig. 2 shows spatial structure of the coaxial undulator periodic magnetic field components, calculated according to the Eq. (1). The longitudinal component of the magnetic field changes sign when the radius \( b > r > a \) is changed and goes to zero approximately in the mid-section of CDC \( r = (a + b)/2 \). The radial component \( H_r \) reaches an extremum at this point (maximum or minimum depending on the longitudinal coordinate).
3. Focusing and Transporting Annular REB by Coaxial Magnetic Undulator

Consider the process of REB transport in CMU based on permanent magnets. The set of equations describing the focusing of REB particles in periodic magnetic undulator includes motion equation in magnetic field (1) and in REB’s proper electromagnetic field. In the approximation of REB piecewise homogeneous density profile:

\[ n(r) = \begin{cases} n_0, & b > r_2 > r > r_1 > a, \\ 0, & r > r_2, \ r_1 > r > a, \end{cases} \]

where \(r_{1,2}\) represent inner and outer radii of the REB, while the expression for the Lorentz force, which acts from the direction of the proper magnetic field reads as:

\[ F_r = -\frac{eE_0}{\gamma^2} Q(\rho), \]

\[ Q(\rho) = \frac{1}{\rho} \left[ Q_0 + \Lambda - \frac{\rho^2 - \rho_1^2}{\rho_2^2 - \rho_1^2} \right], \]

where \(Q_0 = \frac{1}{2} - \frac{\ln \eta}{\eta^2 - 1} - \ln \rho_2, \ \rho = r/a, \ \eta = r_2/r_1, \ \rho_{1,2} = r_{1,2}/a, \ \Lambda = \ln(b/a), \ E_0 = 2I_b/a\Lambda\beta_c, \ I_b\) is the beam current, \(\beta_c = v_z/c, \ v_z\) is the longitudinal velocity component \(\gamma_z = (1 - \beta^2_z)^{-1/2} \). Description of the annular REB focusing by the CMU periodic magnetic field will be hereinafter based on the approximation of the beam envelopes. The reduced motion equation set for annular REB boundaries in this case reads as [6]:

\[ u_{z_i} \frac{dp_{r_i}}{d\xi} - \frac{p_{z_i}}{\gamma_i\rho_i} \frac{1}{\xi} = -\frac{p_{z_i}}{\gamma_i\rho_i} h_z(\rho_i, \xi), \]

\[ u_{z_i} \frac{dp_{r_i}}{d\xi} + \frac{p_{z_i}}{\gamma_i\rho_i} h_z(\rho_i, \xi) = u_{r_i} h_r(\rho_i, \xi), \]

\[ \frac{dp_{r_i}}{d\xi} = \frac{h_r(\rho_i, \xi)}{p_{z_i}}, \]

where \(Q_2 = Q_0, \ p_{1,2} \rightarrow p_{1,2}/mc\) is the reduced momentum, \(\bar{u}_{1,2} = \bar{v}_{1,2}/c, \ \xi = z/a, \ \gamma_{1,2} = \sqrt{1 + p_{1,2}^2}, \ \varepsilon = 2I_b/I_\Lambda\Lambda, \ I_\Lambda = mc^2/\varepsilon = 17\ \text{kA}, \ h_{r,z}(\rho_{1,2}, \xi) = \frac{\epsilon a}{mc^2} H_{r,z}\) are reduced components of the undulator magnetic field (1).

The magnetic field component expressions are treated as infinite Fourier series with respect to spatial harmonics. However, reliable approximation to the real CMU magnetic field distribution accounts only for the first spatial harmonic \(n = 1\).

The set of equations (2) has a motion integral which invokes the law of generalized particle momentum conservation:

\[ p_\varphi = \frac{P_{\varphi0}}{\rho} - \frac{h_w}{q_w} F_1(q_w \rho) \sin(q_w \xi), \]

where \(P_{\varphi0}\) – the original generalized momentum value, \(Q_1 = Q_0 + \Lambda, \ h_w = 2\sqrt{\frac{a\epsilon}{mc^2}} H_0, \ F_1(q_w \rho) \equiv F_1^{(1)}(q_w \rho). \)

Consider that \(P_{\varphi0} = 0\), that is to say that the electron
beam enters into CMU without being gyrated around its own axis. Besides, we shall neglect the variations in the longitudinal beam velocity while it is oscillating in CMU, i.e. \( u_{\perp 1,2} = \beta_0 v_0/c, \) \( v_0 \) – the initial value of the beam’s longitudinal velocity, \( \gamma_{1,2} = \gamma_0 = (1 - \beta_0^2)^{-1/2} \). By using the motion integral (3), the set of boundary particle motion (2) can be reduced as:

\[
\frac{dp_{r1,2}}{d\xi} = \frac{h_w^2}{2\beta_0 \gamma_{1,2} q_w} F_1(q_w \rho_{1,2}) \\
\times \left[ \frac{1}{q_w \rho_{1,2}} F_1(q_w \rho_{1,2}) - F_0(q_w \rho_{1,2}) \right] \\
\times (1 - \cos(2q_w \xi)) - \frac{\varepsilon}{\beta_0^2 \gamma_0} Q_{1,2},
\]

(4)

where \( \gamma_{1,2} = \gamma_0 \sqrt{1 + p_{r1,2}^2 + \xi_{1,2}^2}, \) \( \xi_{1,2} = \frac{h_w}{q_w} F_1(q_w \rho_{1,2}) \sin(q_w \xi), \) \( F_0(q_w \rho) \equiv F_1^{(0)}(q_w \rho). \)

Consider that the longitudinal coordinate-dependent focusing force (averaged over undulator period) compensates for the defocusing force from the direction of the REB proper electromagnetic field. Thus, the following equalities must be attained:

\[
\mu_w F_1(q_w \rho_1) \left[ \frac{1}{q_w \rho_1} F_1(q_w \rho_1) - F_0(q_w \rho_1) \right] = \varepsilon (Q_0 + \Lambda),
\]

\[
\mu_w F_1(q_w \rho_2) \left[ \frac{1}{q_w \rho_2} F_1(q_w \rho_2) - F_0(q_w \rho_2) \right] = \varepsilon Q_0,
\]

where \( \mu_w = \beta_0 \gamma_0 h_w^2/2q_w. \)

Fig. 3 displays the relationships of annular REB optimum radii as a function of the undulator magnetic field strength \( H_0 \) at fixed values of the beam current \( I_b = 3 \) kA, CMU period \( L_w = 3.92 \) cm and geometrical dimensions of CDC \( a = 2 \) cm, \( b = 4 \) cm. With increasing of the undulator magnetic field, its inner radius increases as well, while its outer radius decreases. Fig. 4 shows the REB envelopes produced through the numeric solution of the equation set (4) for the values of the magnetic field strength \( H_0 = 3 \) kOe and current \( I_b = 3 \) kA. Initial values of the beam boundary radii coincide with the equilibrium ones in accordance with Fig. 3. Visibly, the internal and external envelopes of the annular REB perform radial oscillations, while the beam itself as a whole rears periodically up, its thickness remaining practically unchanged in the meantime. The radial velocities of the boundary particles are small. In the azimuthal direction, against the background of beam gyration on the whole, boundary particles velocity exhibits intensive harmonic oscillations with the spatial period being equal to the undulator period.

4. Microwave Amplification in Coaxial Ubitron

Consider the process of microwave amplification in coaxial Ubitron by thin annular REB. The undulator of this beam electrons configuration exhibits oscillations in the azimuthal direction, and for this reason REB propagating in the coaxial undulator should excite electromagnetic waves of TE-type. Dependence of the longitudinal wave numbers on the frequency of symmetrical waves of this type can be viewed as:

\[
k_n(\omega) = \sqrt{\frac{\omega^2}{c^2} - \frac{\Lambda_n^2}{a^2}},
\]
where $\lambda_n$ are the transcendental equation roots $J_1(\lambda_n s)N_1(\lambda_n) - J_1(\lambda_n)N_1(\lambda_n s) = 0$, $s = b/a$. In particular, for $s = 2$, $\lambda_1 = 3.24$, $\lambda_2 = 6.31$.

The electron beam will be in parametric synchronism with electromagnetic wave, provided the following condition is met:

$$\omega = (k_n(\omega) + k_w)v_0. \quad (5)$$

The relation (5) determines the frequencies of phase-locked electromagnetic waves $TE_{0n}$ in the coaxial waveguide

$$\omega = k_w v_0 \gamma_0^2 \left(1 \pm \sqrt{\beta_0^2 - \frac{\lambda_n^2}{k_w^2 a^2 \beta_0^2}}\right).$$

The index "+" stands for forward waves and the one "−" for backward waves. The synchronism of REB with its eigenwaves is feasible, when

$$k_w^2 a^2 \beta_0^2 > \lambda_n^2.$$

Obviously, a finite number of radial harmonics can be in the synchronism with the beam, while in order to meet the condition

$$\lambda_2 > k_w a \gamma_0 \beta_0 > \lambda_1$$

there is only the fundamental harmonic $n = 1$ to do it.

The process of the symmetrical TE-wave amplification in coaxial waveguide by electron beam, we shall use the inhomogeneous wave equation for the electric field component $E_\varphi$:

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial E_\varphi}{\partial r} = \frac{1}{r^2} E_\varphi + \frac{\partial^2 E_\varphi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E_\varphi}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial j_\varphi}{\partial t}, \quad (6)$$

where $j_\varphi$ is the azimuthal current which is determinable by the REB particle distribution function.

The equation (6) will be sought as

$$E_\varphi = \frac{1}{2} \Phi_n(r) \left[A_n(z)e^{ik_n z - i\omega t} + k.c.\right], \quad (7)$$

where $\Phi_n(r) = J_1(\lambda_n r/a) - \frac{J_1(\lambda_n)}{N_1(\lambda_n)} N_1(\lambda_n r/a)$ is the function describing the radial structure of the axially-symmetrical eigenwave with the wave number $n$, $A_n(z)$ is the slowly changing amplitude of the wave being amplified. Substituting the expression for $E_\varphi$ (7) in the wave equation and averaging over the wave period $T = 2\pi/\omega$ and cross-section of the coaxial waveguide (while taking into account the orthogonality of the functions $\Phi_n(r)$), we obtain the following reduced equation for the amplified wave amplitude:

$$\frac{dA_n}{dz} = \frac{4\pi}{c^2} k_n < j_\varphi >, \quad (8)$$

where

$$< j_\varphi > = \frac{1}{N_n T} \int_a^b \int_0^T rdr\Phi_n(r) dt j_\varphi e^{i\omega t - ik_n z}, \quad (9)$$

$N_n = \int r dr \Phi_n^2(r)$ represents the norm of orthogonal functions $\Phi_n(r)$. The equation (9) can be transformed as follows:

$$< j_\varphi > = \frac{1}{N_n T} \int_a^b \int_0^T dte^{i\omega t - ik_n z} \int_{-\infty}^{\infty} dp dpz v_\varphi f(p, p_z, \alpha, r, z), \quad (10)$$

where $f(p, p_z, \alpha, r, z)$ is the REB particle distribution function, $p_\perp, p_z, \alpha$ are the momentum coordinates in the cylindrical coordinate system in the momentum space. Using the phase volume conservation law, the expression (10) can be transformed into the following form:

$$< j_\varphi > = \frac{2\pi e c}{N_n T} \int_a^b \int_0^T r dr_0 \int_0^\infty dp_\perp dp_z v_\varphi f(p_\perp, p_z, \alpha, r, z) \int_{-\infty}^{\infty} dp_{\varphi \perp} \Phi_n(r_{\varphi \perp}) e^{i\omega t - ik_n z} f_0(p_{\perp 0}, p_z, r_0) \frac{v_\varphi L}{v_{\varphi \perp}}. \quad (11)$$

Here $f_0(p_{\perp 0}, p_z, r_0) = f(p, p_z, \alpha, r, z = 0)$ is the REB distribution function in the injection plane $z = 0$, $t_L$ – the electron in-flight time (Lagrangian time), $r_L$ – the Lagrangian radial coordinate of electron, $v_{\varphi \perp}, v_{\varphi L}$ – the Lagrangian component velocities. The original distribution function will be chosen such as

$$f_0(p_{\perp 0}, p_z, r_0) = \frac{I_b}{v_0} \frac{\delta(r_0 - r_b) \delta(p_{\perp 0})}{2\pi p_0} \frac{\delta(p_{\perp 0} - p_0)}, \quad (12)$$

$p_0$ is the initial particle beam momentum, $r_b$ is the beam radius. The distribution function (12) describes an infinitely thin annular flux of unexcited phase-unlocked oscillators. For this kind of averageable current, the expression (11) is simplified drastically:

$$< j_\varphi > = \frac{I_b}{2\pi N_n T} \int_0^T dt_0 \frac{v_{\varphi L} \Phi_n(r_{\varphi L}) e^{i\omega t - ik_n z}}{v_{\varphi \perp}}. \quad (13)$$

Brigin out the slow phase in the formula (13) required to substitute the azimuthal electron oscillation velocity in the undulator magnetic field which follows from the expression (3) in the right-hand
side of the Eq. (13). As a result, we shall obtain the following:

\[
\langle j_{\varphi} \rangle = \frac{i}{4\pi} \frac{I_b h_w}{N_n F_{n0}} \int_{0}^{2\pi} d\theta_0 e^{i\theta} \frac{1}{p_z} \times \Phi_n(r_L) F_1(k_w r_L),
\]

where \( \theta = \omega t_L - (k_n + k_w) z \) is the phase coordinate of particle relative to the ponderomotive force, \( \theta_0 = \omega t_0 \), \( p_z = p_{z0} / mc \). The excitation equation (8) will then be:

\[
\frac{dC_n}{d\xi} = -i \mu \frac{1}{2\pi} \int_{0}^{2\pi} d\theta_0 e^{i\theta} \frac{1}{p_z} \Phi_n(r_L) F_1(k_w r_L), \tag{14}
\]

where \( \mu = \frac{\omega_0^2 I_b}{k_n c N_n I_A} \), \( h_w = \frac{2\sqrt{2} e H_0 a}{\pi mc^2} \).

The excitation equation (14) must be supplemented with the averaged equations of beam particle motion in the fields of the electromagnetic wave and magnetic undulator, as well as with those for this motion in the electromagnetic microwave field of the REB space charge.

\[
p_z \frac{dp_z}{d\xi} = -i \frac{h_w}{4q_w p_0} \Phi_n(\rho) F_1(q_w \rho) C_n e^{-i\omega},
\]

\[
p_z \frac{d\gamma}{d\xi} = -i \frac{h_w}{4q_w} \Phi_n(\rho) F_1(q_w \rho) C_n e^{-i\omega},
\]

\[
p_z \frac{dp_z}{d\xi} = i \frac{h_w}{4q_w} \frac{1}{p_0} \left[ \lambda_n \Phi_n(\rho) F_1(q_w \rho) \right] C_n e^{-i\omega} + k.c.,
\]

\[
\frac{d\theta}{d\xi} = \frac{q_0}{p_z} \left( \frac{\gamma}{p_z} - \frac{\gamma_0}{p_0} + \Delta f \gamma_0 \right),
\]

\[
\frac{d\rho}{d\xi} = \frac{p_r}{p_z},
\]

where \( \rho_\omega = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i\omega} d\theta \).

Following reduced variables and parameters are used in the set of equations (15). The radial coordinate of the particle of the beam with the entering phase \( \theta_0 = \omega t_0 / \omega \), \( \Delta f = \Delta \omega / \omega \) is the relative frequency detune, \( \Delta \omega = \omega - (k_n + k_w) \omega_0 \), \( q_0 = \omega a / c \), \( q_n = k_n a \), \( \gamma \) is the relativistic factor of particle, \( \gamma_0 \) is its initial value, \( p_0 = \beta_0 \gamma_0, \beta = q_0 / p_0 \).

\[
Q = [J_0(q_w \rho_b) K_0(\gamma) - I_0(\gamma) K_0(q_w \rho_b)] \\
\times \left[ \frac{I_0(q_w \rho_b) K_0(\gamma_0) - I_0(\gamma) K_0(q_w \rho_b)}{I_0(\gamma) K_0(\gamma) - K_0(\gamma) I_0(\gamma_0)} \right], \tag{16}
\]

The first component in the right-hand side of the longitudinal momentum equation set (15) describes the ponderomotive force from the direction of the undulator and electromagnetic wave, while the second one is taking into account the impact of the REB microwave spatial charge.

The following integral ensues from the first two equations of the set (15)

\[
\gamma - \beta_0 p_z = 1, \tag{17}
\]

that connects the energy and longitudinal momentum of the particles.

Define the efficiency of the amplifier as a wave power-to-initial REB power ratio. In the nondimensional variables chosen, the amplifier efficiency can viewed as follows:

\[
\eta = \frac{I_A}{4I_b} \frac{k_n N_n}{k_0 (\gamma_0 - 1)^2 |C_n|}. \tag{18}
\]

### 4.1. Linear Amplification Theory

Provided the annular REB is placed in the middle of the coaxial drift chamber \( r_s = (a+b)/2 \) the radial force in (15) is low and the radial REB particle displacement can be ignored. In this event, the sets of equations (14), (15) are simplified and assume the form:

\[
\frac{dC_n}{d\xi} = -i \mu G \frac{1}{2\pi} \int_{0}^{2\pi} d\theta_0 \frac{e^{i\omega}}{p_z},
\]

\[
p_z \frac{dp_z}{d\xi} = -i \frac{h_w}{4q_w \beta_0} GC_n e^{-i\omega},
\]

\[
p_z \frac{d\gamma}{d\xi} = -i \frac{h_w}{4q_w} \frac{1}{p_0^2} \frac{Q_0}{I_A} e^{-i\omega} p_\omega + k.c.,
\]

\[
p_z \frac{dm}{d\xi} = -i \frac{h_w}{4q_w} \frac{1}{p_0^2} \frac{Q_0}{I_A} e^{-i\omega} p_\omega + k.c., \tag{18}
\]

\[
\frac{d\theta}{d\xi} = \frac{q_0}{p_z} \left( \frac{\gamma}{p_z} - \frac{\gamma_0}{p_0} + \Delta f \gamma_0 \right) + \Delta,
\]

where \( \Delta = \frac{q_0}{\beta_0} \Delta f, G = \Phi_n(\rho_b) F_1(q_w \rho_b) \).

For small amplified waves amplitudes the equation set (18) can be linearized. As a result, we have the following bound equations set:

\[
\frac{d^2 C_n}{d\xi^2} + i \Delta \frac{dC_n}{d\xi} - \Pi AC_n \]

\[
= \Pi \left( B_{n_0} - i \frac{dn_\omega}{p_0} \frac{d\xi}{d\xi} \right), \tag{19}
\]

\[
\frac{d^2 n_\omega}{d\xi^2} + K_0^2 n_\omega = -\frac{q_0}{\gamma_0} AC_n.
\]
where \( C_\omega = C_n e^{i\Delta \xi}, n_\omega = \rho_\omega e^{i\Delta \xi}, \Pi = \mu G, A = \frac{h_w}{4q_0 p_0 \gamma_0} Q, B = \frac{q_0 I_b}{p_0 I_A} Q, K_b^2 = \frac{q_0}{p_0 \gamma_0} B. \)

Equation set solution (19) will be sought as follows:

\[
C_n = C_{n0} e^{i\Gamma \xi}, \quad n_\omega = n_{\omega0} e^{i\Gamma \xi}.
\] (20)

Substituting (20) into equation set (19) yields the following characteristic equation:

\[
(\Gamma + \Delta)(\Gamma^2 - K_b^2) + \Pi A \left( \Gamma + \frac{q_0}{p_0 \gamma_0} \right) = 0.
\] (21)

In case of REB’s weak current, while meeting the conditions of the precise synchronism, the following increment expression follows from the dispersion equation:

\[
\delta = \sqrt{\frac{3}{2a}} \left( \frac{\Pi A q_0}{p_0 \gamma_0} \right)^{1/3}.
\] (22)

The increment (22) is proportional to the beam current to power of 1/3 and describes the Compton regime of the microwave amplification.

In the general case, the dispersion equation (21) is solved by the numerical methods. Fig. 5 presents increment as a function of detuning \( \Delta \) for various beam values. One can well see that the increment reaches the maximum value for all considered currents when the detuning is negative \( \Delta \). Upon reaching its maximum with increasing de-tuning \( \Delta \), the increment falls fast and goes to zero. Fig. 6 presents the dependence of maximum increment on beam current. For small currents the increment is increasing fast, while for large ones the growth slows down. With an increasing magnetic field strength of the coaxial undulator, the increment rises monotonously.

### 4.2. Results of Non-linear Theory of Amplification in Coaxial Ubitron

The set of nonlinear equations (18) was solved using numerical techniques for various values of the annular REB parameters and coaxial magnetic undulator. Geometric parameters of the coaxial drift chamber, beam energy, its mean radius and undulator period were fixed. Fixed parameters were as follows: inner radius of CDC \( a = 2 \) cm, its outer radius \( b = 4 \) cm, the energy \( U = 490 \) kV, the mean tubular beam radius \( r_b = 3 \) cm, the undulator period \( L_w = 3.92 \) cm. For those parameters the maximum amplification occurs at the frequency \( f = 7.76 \) GHz. The beam current and undulator magnetic field strength varied during this study.

Fig. 7 presents the relationship of the wave power
5. Nonlinear Dynamics of Ubitron Oscillator with Delayed Feedback

Microwave oscillator with delayed feedback contains a nonlinear coaxial Ubitron amplifier as described in the preceding Section, and a linear circuit of the external feedback which serves to feed signal from amplifier output to its input.

The delayed-feedback oscillator parameters is described by a set of nonlinear equations which contains a transient state equation for the electromagnetic wave amplitude:

\[
\frac{\partial C}{\partial \tau} + \frac{\partial C}{\partial \xi} = -i\mu G \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} \frac{d\theta}{p_2} \quad (23)
\]

and electron motion equations within the Lagrangian variables:

\[
p_\xi \frac{dp_\xi}{d\xi} = -i \frac{h}{2q_0} G C e^{-\theta} - i \frac{q_0 \gamma_0}{p_0} I_b Q p_\omega e^{-i\theta} + k.c., \quad (24)
\]

\[
p_\xi \frac{d\gamma}{d\xi} = -i \frac{h}{2q_0} G e^{-\theta} - i \frac{q_0}{p_0} I_b Q p_\omega e^{-i\theta} + k.c., \quad (25)
\]

\[
\frac{d\theta}{d\xi} = q_0 \left( \gamma - \frac{\gamma_0}{p_0} \right), \quad (26)
\]

where \( \tau = \frac{v_g}{a} \left( 1 + \frac{v_g}{v_0} \right)^{-1} (t - z/v_0) \), \( v_g \) is the group wave velocity.

Self-contained set of equations (23)–(26) needs to be complemented with the initial conditions for the motion equations:

\[
\theta(\xi = 0) = \theta_0, \quad \left. \frac{d\theta}{d\xi} \right|_{\xi = 0} = 0 \quad (27)
\]

and the delayed feedback equation for the wave amplitude

\[
C(\tau, 0) = R e^{i\Delta\varphi} C(\tau - T_d, l), \quad (28)
\]

where \( R \) is the transmittance-amplitude coefficient, \( \Delta\varphi \) is the signal phase shift in the feedback loop, \( l = L/a, L \) is the oscillator length, \( T_d = \frac{v_0}{v_0/v_g - 1} \), \( v_g \) is the group velocity in the feedback loop.

From the equations sets (23)–(25) a relation ensues:

\[
|C(\xi = l, \tau)|^2 - |C(\xi = 0, \tau)|^2 + \frac{1}{l} \int_0^l |C(\xi, \tau)|^2 d\xi + \Lambda \frac{1}{2\pi} \int_0^{2\pi} (\gamma(\xi = l, \tau, \theta_0) - \gamma_0) = \text{const}, \quad (29)
\]

which brings on the energy conservation law: \( \Lambda = 4\omega a^2 l b \).

The first two terms in the left-hand side of the equation (29) account for the difference of the microwave energy fluxes at oscillator’s input and output, the third term considering energy variations throughout the oscillator volume. During the steady state operation, this term goes to zero. And, finally, the last term is the difference of fluxes of the REB kinetic energy at output and input of the oscillator.

5.1. The Oscillator Starting Currents

Starting currents of Ubitron oscillator were determined at the following fixed parameters: coaxial line inner radius \( a = 2 \text{ cm} \), its outer radius \( b = 4 \text{ cm} \), normalized to the initial beam power (efficiency) as a function of the longitudinal coordinate for various currents. At short distances, the wave power increases exponentially, reaching its maximum value before oscillating due to the phase oscillations of the bunches in the potential well of the ponderomotive potential. With increasing current, the distance to the first (main) wave power maximum becomes shorter. Fig.8. presents the relationship of the amplifier efficiency in the first maximum as a function of beam current. With increasing beam current, the beam efficiency follows reaching as high as 35 %. The efficiency rises also with the increase of the undulator magnetic field strength.

Fig. 8. The amplifier efficiency vs. beam current. \( a = 2 \text{ cm}, b = 4 \text{ cm}, \ L_w = 3.92 \text{ cm}, 2 - H_0 = 3 \text{ kOe} \).
Fig. 9. Dependence of starting current on feedback generator length: 1 – $H_0 = 2$ kOe, 2 – $H_0 = 3$ kOe, 3 – $H_0 = 4$ kOe. Parameters of the oscillator: $a = 2$ cm, $b = 4$ cm, $r_b = 3$ cm, $R = 0.3$, $\Delta \varphi = 0$, $L_w = 3.92$ cm.

Fig. 10. Starting current as a function of undulator magnetic field. 1 – $L = 30$ cm, 1 – $L = 50$ cm, 1 – $L = 70$ cm. Generator parameters: $a = 2$ cm, $b = 4$ cm, $r_b = 3$ cm, $L_w = 70$ cm, $H_0 = 3$ kOe, $\Delta \varphi = 0$.

Fig. 11. Starting current as a function of transmittance $R$. Generator parameters: $a = 2$ cm, $b = 4$ cm, $r_b = 3$ cm, $L_w = 70$ cm, $H_0 = 3$ kOe, $\Delta \varphi = 0$.

Fig. 12. The amplitude of oscillations $|C(l, \tau)|$ at the oscillator output end as a function of time $I_b = 0.2$ kA.

The amplitude of oscillations $|C(l, \tau)|$ decreases with increasing transmittance $R$ and at the value of $R = 1$ it goes to zero.

5.2. Computational Results

The numeric solutions of the equation sets (23)–(26) with the initial and boundary conditions (27),(28) were found for the mentioned in the above Section parameters of the magnetic undulator and REB, undulator magnetic field strength $H_0 = 3$ kOe, $L_w = 70$ cm, feedback loop parameters $R = 0.3$, $\Delta \varphi = 0$ and various beam current values. Numeric calculations indicated that depending on the electron beam current, Ubitron oscillator demonstrates different modes of microwave generation. At the electron beam current $I_b = 0.2$ kA steady-state generation mode is present. Fig. 12 shows the relationship for the current on field transmittance in the feedback loop.
amplitude of the wave $|C|$ at the output end of Ubitron oscillator ($\xi = l$) vs. time. One can well see that at the linear generation stage the amplitude of the wave rises monotonously over time [10]. The observable wave amplitude oscillations can be attributed to the inhomogeneity of the field distribution along the coaxial system on account of the beam presence. This kind of inhomogeneous field distribution with a clear-cut minimum circulates from output to input which is the main reason for the non-monotonous growth of the wave amplitude with time. At the nonlinear stage, following the transient process, the amplitude reaches its stationary value. In the steady-state regime a non-symmetrical distribution of the amplitude occurs along the system. With an increasing beam current $I_b > 0.225$ kA, the steady-state mode of oscillation is disrupted and the system sees the regular automodulation regime establishment. The frequency spectrum of amplitude oscillations in the logarithmic scale is derived as:

$$S(\omega) = \lg \frac{S(\omega)}{S_{\text{max}}} + 4,$$

where $S(\omega)$ – the frequency spectrum, $S_{\text{max}}$ – the maximum value of $S(\omega)$. For the current $I_b = 0.3$ kA established self-modulated oscillations were
actually harmonic. Current increase up to 0.5 kA brings on a more complicated regulation of the self-modulation (Fig. 13). The frequency spectrum contains intense multiple harmonics. Fig. 14, 15 illustrate the dynamics of microwave generation for the beam current $I_b = 0.7$ kA. Bifurcations of a double period are observed. The spectrum is enriched with multiple harmonics and, at the current $I_b = 0.775$ kA, chaotic self-modulation mode comes on strong Fig. 16,17. The frequency spectrum has a solid curve against the backdrop of which there are some peaks that correspond to multiple harmonics. The autocorrelation function decays quickly initially, followed by oscillations that are accounted for by the curves in the frequency spectrum. We shall dwell a little on the issue of the oscillator efficiency, restricting to the steady-state generation regime. We shall define the oscillator efficiency $\eta$ as the extracted microwave-to-initial beam power ratio. Defined likewise the efficiency should be as follows:

$$\eta = \frac{1-R^2}{\lambda(\gamma_0-1)}|C(\xi = l)|^2.$$ 

With an increasing electron beam current, the efficiency rises as well. So, at the current 0.1 kA it is 13.2 %, at $I_b = 0.15$ kA, $\eta = 21.1$ %, and with the increase of the current up to 0.2 kA the efficiency becomes as high as 23.4 %.

6. Coaxial Ubitron Backward Wave Oscillator Dynamics

The section addresses the nonlinear dynamic of Ubitron oscillator, driven by backward electromagnetic wave. The electron beam is in parametric synchronism with the backward wave, provided that the following condition is met:

$$\omega = (k_w - k_n(\omega))v_0.$$  

Nonlinear dynamics of Ubitron backward wave oscillator (BWO) contains the non-steady state equation of excitation of the backward electromagnetic wave:

$$\frac{\partial C}{\partial \tau} - \frac{\partial C}{\partial \xi} = -i\mu G \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}}{p_x} d\theta_0$$  \hspace{1cm} (30)

and motion equations within the Lagrangian variables (24)–(26). In the equation (30) the non-dimensional time is determined in the following way:

$$\tau = (v_y/a)(1 + v_y/v_0)^{-1}(t - z/v_0).$$

The self-consistent set of equations (30), (24)–(26) should be complemented by the initial conditions for the motion equations

$$\theta(\xi = 0) = \theta_0, \quad \left. \frac{d\theta}{d\xi}\right|_{\xi=0} = 0 \quad (31)$$

and boundary and initial condition for the wave amplitude

$$C(\tau, l) = 0, \quad C(0, \xi) = C_0, \quad (32)$$

where $l$ is the reduced length of REB coupling area with the backward wave, $C_0$ is the original amplitude value.

Starting current $I_{st}$, at which the self-excitation of Ubitron oscillator occurs, as a function of its length is shown in Fig. 18. The above relationship was obtained for the following parametric values of the oscillator: inner radius of the coaxial line section $a = 2$ cm, outer radius $b = 4$ cm, undulator period $L_w = 2.59$ cm, beam energy $U = 490$ keV, beam radius $r_b = 3$ cm, CMU magnetic field strength $H_0 = 3$ kOe. With an increasing length of the autooscillator, starting current becomes smaller.

Numeric solutions of the nonlinear set of equations (30), (24)–(26) with the initial and boundary conditions (31), (32) were obtained for a fixed value of the oscillator length $L = 70$ cm. The beam current varied within the limits $10.3$ kA $\geq I_b \geq 1.2$ kA. For the above CMU parameters and REB energy the generation frequency was 7.94 GHz, the starting current being $I_{st} = 1$ kA.

Numeric calculations indicated that, depending on the beam current, the oscillator demonstrates different microwave generation modes. In order to illustrate different microwave generation modes, we obtained the relationships of the amplitude of oscillations at the input end of the oscillator $|C(\xi = 0, \tau)|$ and frequency spectra on the logarithmic scale of those relationships.

![Fig. 18. The Dependence of Starting Current on Oscillator Length.](image-url)
For the currents range $1.3 \, \text{kA} \geq I_b \geq I_{st}$ the process of the steady-state oscillation establishes. This mode of operation is illustrated in Fig. 19. Any further current increase leads to disruption of the steady-state generation and establishing self-modulation regime. Fig. 20 shows the dependence of amplitude at the oscillator input on REB current time $I_b = 1.5 \, \text{kA}$. Following the linear phase of the
exponential growth in the settled mode the amplitude exhibits periodic oscillations, which by the look of them are close to being harmonic. The regime of oscillations self-modulation is disrupted at the REB current $I_b = 2.64$ kA, the steady-state mode restored. Fig. 21 illustrates the steady-state mode for the current $I_b = 2.8$ kA. Further on, for the current $I_b \geq 2.9$ kA, the regime of self-modulation of the amplitude of oscillations establishes, while with the increasing beam current the depth of self-modulation becomes greater and the amplitude-time curve gets more complicated. Fig. 22 presents the amplitude-time curve and Fig. 23 shows the frequency spectrum for the current $I_b = 4.3$ kA. Evidently, the self-modulation regularity, while remaining periodic, is of a very intricate nature, which is proved by the spectrum of amplitude oscillations. With any further increase of the beam current from $I_b = 4.3$ kA up to $I_b = 10.3$ kA the nature of self-modulation gets simpler to assess. At the beam current $I_b = 10.3$ kA (Figures 24, 25), the amplitude-time relationship is much easier to handle, while the spectrum contains the multiple harmonics.

We shall now address the problem of the oscillator efficiency in relation to the steady-state generation mode. For the above numeric parameters of the oscillator, we can derive a simple efficiency formula: Efficiency $= 2.25 \left| C \right|^2 / I_b$(kA).

Consequently, for the current $I_b = 1.2$ kA the efficiency is 9.4 %, while by increasing the current up to $I_b = 2.8$ kA the efficiency gets as high as 13 %.

7. Conclusions

Coaxial Ubitron amplifier operation theory with the undulator based on circular permanent magnets that are magnetized longitudinally is constructed. The structure of coaxial undulator magnetic fields is studied. The transport of high-current annular REB in the periodic magnetic field of this undulator is considered in the approximation of envelopes and the conditions for the optimum REB transport are formulated.

The linear theory is constructed for the -type microwaves amplification in the coaxial Ubitron. Using numeric solution of the dispersion equation we obtained the relationships of increment and beam current. It was demonstrated that with an increasing current the increment rises monotonously. Nonlinear stage of microwave amplification in the coaxial Ubitron is constructed. The relationships are derived for the Ubitron efficiency as a function of beam current and magnetic field strength. For an Ubitron with the parameters: inner radius of the coaxial $a = 2$ cm, outer radius $b = 4$ cm, undulator period $L_u = 3.92$ cm, beam energy $- U = 490$ kV, beam current $- I_b = 3$ kA, mean beam radius $- r_b = 3$ cm, magnetic field strength $- H_0 = 3$ kOe, the efficiency is 30 %, the output power output being 440 MW. The Ubitron length is no greater than 66 cm.

Nonlinear dynamics was studied for microwaves generation in the coaxial Ubitron oscillator with external delayed feedback. Derivation of a self-contained nonlinear set of equation depicted the process of generation of electromagnetic oscillations in this oscillator. As a first step in the theory, we studied the relationships of starting current as a function of oscillator length and undulator magnetic field strength. With increasing values of the said parameters, the generation starting current decreases. Qualitative nature of microwave generation in REB-driven oscillators is determinable by its current. The oscillator behavior was studied in regard to beam-current variations. At a relatively weak current, steady-state microwave generation regime is realizable. At higher values of the bifurcation-determinable beam current $I_b = 0.3$ kA the steady-state operation mode of generation is disrupted and the system allows the regime of steady-state self-modulation. This period is followed by the double period bifurcation and, in the long run, the regime of random oscillations establishes. In this chaotic regime the frequency spectrum against the background of the continuous component contains a number of peaks. In the steady-state operation mode, at the value of the current $I_b = 0.2$ kA the efficiency is 23.4 %.

The study also addresses the nonlinear dynamics of microwave generation in coaxial Ubitron oscillator driven by the backward wave. Qualitative nature of microwaves generation in REB-driven Ubitron oscillator is determinable by its current. The oscillator behavior is studied against the increase of the beam current. It is demonstrated that with increasing current in the studied range of $1.2$ kA $\leq I_b \leq 10.3$ kA, the microwave generation dynamic changes dramatically. For small currents the current the said
interval includes generation of steady-state mode oscillations, which is then replaced with the self-modulation operation mode. This pattern repeats itself further on. The regime of steady-state generation establishes again, just to be replaced with the self-modulation mode. Note that with the rising current the self-modulation pattern becomes more complicated, at first, only to become simplified later. In the steady-state regime, with increasing REB current, the efficiency of the microwave power generation increases.

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References


