Cantor Distribution

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Contents

Abstract

To build the fractal electro-dynamics theory, a new distribution is introduced for consideration whose construction is based on the Cantor set structure and Cantor measures. Major properties of this Cantor distribution are discussed. Semigroup of operators assigned on the Cantor measures is considered. As an example, a theory is brought up for consideration of the permittivity dissipative burst transition scattering on a charge at rest.

1. Introduction

The subject of fractal medium electrodynamics studies electromagnetic processes in a space, filled up by such a substance that constitutes the fractal structure. As any macroscopic theory, the fractal medium electrodynamics operates such physical values that are averaged over "physically infinitely small"volume fractal components without going into the atomic-molecular structure of the matter. Fractal structures include fractal clusters, fractal surfaces, percolation clusters and other material entities that are observable at present in experiments. An especially important moment in constructing the fractal electrodynamics is the procedure of averaging, the construction of which is possible, applying fractal and multifractal measures.

In this way, at the foundation of construction of the fractal medium electrodynamics is the fundamental notion of the fractal and multifractal, to be more precise, the Cantor set structure and the fractional dimensionality space. There are several directions, in which the fractal electrodynamics mathematical apparatus is basically pushed forward toward the new frontiers. For example, a major consideration is given to the fractional integral-differentiation and fractal wavelets, or the iteration functional systems are taken as a basis. This paper considers the generalized functions (distributions) apparatus as underlying mathematical base of the fractal electrodynamics theory. This approach is most general, from my view-point, containing particular cases, fractional integral differentiation and wavelets.

As an example, consideration is given to the fractal spectrum formation (the Cantor set point radiation emission frequency localization) during transition scattering. The permittivity wave is assigned in the meantime by the dissipative burst function $\gamma_\xi(x)$. A rather complicated analytical form of the dielectric burst can be brought into existence by shock waves running through dielectric material, by turbulent movements in plasma-like media and the fluids.

2. Cantor Set

The construction of the classic triad Cantor set with the relation ξ is well-known. While constructing it, the unit segment $E_0 = [0,1]$ is divided into three parts and rejection is made of the mean open interval with the value $1-2\xi$. Thus, such a set E_1 is obtained that consists of two closed segments with the value ξ. Then, this procedure is reiterated with two segments $[0,\xi]$ and $[1-\xi,1]$, constituting the set E_1, etc. At the n-th step, such a set E_n is obtained that consists of $2n$ segments of the length ξ^n. The compact set E_n is known as pre-Cantor set, with the Cantor set proper being the intersection of the pre-Cantor sets $E = \bigcap\limits_{n=0}^{\infty} E_n$. (Fig. 1).

The Hausdorf-Besikovich dimensionality of the Cantor set E is equal to $d_H = \ln 2/|\ln\xi|$, $0 < \xi < \dfrac{1}{2}$.

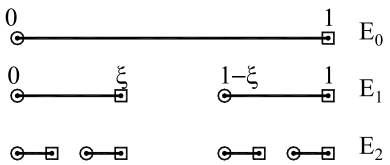

Fig. 1. Pre-Cantor Sets.

The Cantor set is characterized by points of the first and second kinds. The points of the set E that are the extremities (left- or right-hand) of complementary intervals to the Cantor discontinuum are called the Cantor set first kind points (left-hand E^L or right-hand E^R, respectively). Similar sets E_n^R and E_n^L well exist in any of the pre-Cantor sets E_n. In Fig. 1, the points belonging to the set E_n^R are marked by the circles, with those belonging to the set E_n^L, by the squares. These points are naturally ("left to right") ordered, with the Cantor set first kind points forming the countable set. All the remaining points of the set are called points of the second kind. The entirety of all the points of the second Cantor set kind resembles the surd set and has the power of continuum [1]. Any point of the Cantor set E with the relation ξ can be written as:

$$t = \sum_{n=1}^{\infty} \omega_n \xi^n, \qquad (1)$$

where $\omega_n = \left\{0, \xi^{-1}-1\right\}$ takes on only two different values. The numbers $t \in E$ in which all ω_n, beginning from a certain one, are equal to one another, correspond to the Cantor set first kind points.

In this way, the multitude of right-hand points of the Cantor set first kind E_n^R is determined by the expansion (1), all the coefficients of which are zero, beginning from the term $n+1$: $\omega_{n+1} = \omega_{n+2} = \cdots = 0$.

The multitude of left-hand points of the Cantor set first kind E_n^L is determined by the expansion (1), in which all the coefficients, beginning from $n+1$ are equal to one another and to $\xi^{-1}-1$: $\omega_{n+1} = \omega_{n+2} = \cdots = \xi^{-1}-1$.

In order to enumerate all points of the set, for example E_n^R, it is necessary to add two values 0 and $\xi^{-1}-1$ to each ω_k, $k = 1,2\ldots n$. In this way, we shall obtain 2^n points of the set E_n^R, i.e. $N(E_n^R) = 2^n$. It is understood that $N(E_n^R) = N(E_n^L)$.

3. Free Semigroup

The triad Cantor set with the relation ξ is produced by the alphabet A, consisting of two letters $A = \{a, \overline{a}\}$ (one can regard $\overline{a}$ as negation of). This is accounted for by the fact that the mutually unequivocal correspondence represents $a \to 0$, $\overline{a} \to \xi^{-1}-1$ and any word $\alpha = a_1, a_2, \ldots, a_k, \ldots$ from the set of (finite and infinite) words Ω corresponds to just the only Cantor set point t_α. Such an approach that employs the word space is often used in the theory of fractals and multifractals [5], with any a_k that is included in the word α being equal either to or $\overline{a}$. In this way $a_k \to \omega_k$: and

$$t_\alpha = t(\alpha) = \sum_{k=1}^{\infty} \omega_k \xi^k. \qquad (2)$$

For instance, the word $\alpha = a\,\overline{a}\,a\,\overline{a}\ldots(a\,\overline{a})$ for which the combination $(a\,\overline{a})$ is repeated all the time, corresponds to the Cantor set $t_\alpha = \sum\limits_{k=1}^{\infty}(\xi^{-1}-1)\xi^{2n} = \dfrac{\xi}{1+\xi}$. For $\xi = \dfrac{1}{3}$, $t_\alpha = \dfrac{1}{4}$.

In another case, the word $\beta = \overline{a}\,a\,\overline{a}\,a\,\overline{a}\,a\ldots(\overline{a}\,a)$ corresponds to the Cantor set $t_\beta = \dfrac{1}{1+\xi}$. For $\xi = \dfrac{1}{3}$, $t_\beta = \dfrac{3}{4}$.

The numbers 1/4 and 3/4 are known to be the Cantor set second kind points [2]. Really, in the above examples both in the word α and in the word β a combination of two letters, not of one, from the alphabet A is repeated ad infinitum that might as well correspond to the first kind points.

The word sets Ω can be broken down into word sub-sets of the finite length Ω_n, $n = 0,1,2,\ldots$. The number of elements $N(\Omega_k)$ in the set of different words of the length k is equal to $2k$.

The element Ω_0 is the empty word Λ.

Two words $x_1 x_2 \ldots x_m$ and $y_1 y_2 \ldots y_n$ are considered to be equal if they coincide as the sequences $m = n$, $x_1 = y_1$, $x_2 = y_2$, $\ldots$, $x_m = y_m$.

Note too that the bijective mapping $t : \Omega_n \to E_n^R$ puts in correspondence to the word of the length n the pre-Cantor set point $E_n^R; \alpha \to t_\alpha$, $\alpha \in \Omega_n$, $t_\alpha \in E_n^R$.

Using the set Ω of all finite length words in the alphabet A, one can determine the operation of insertion or concatenation (*):

$$x_1\,x_2\,\ldots\,x_m * y_1\,y_2\,\ldots\,y_n = x_1\,x_2\,\ldots\,x_m\,y_1\,y_2\,\ldots\,y_n. \quad (3)$$

In this way, for the two words $\alpha_1 \in \Omega_n$ and $\alpha_2 \in \Omega_n$ (of the length n and k), as a result of the operation of concatenation, we obtain the word of the length $n+k$: $\alpha_1 * \alpha_2 = \alpha_1\alpha_2 \in \Omega_{n+k}$.

The operation of concatenation is associative according to its definition. In this way, the set of finite words Ω, with the operation of concatenation inserted in it, forms the free semigroup $\Omega^* = (\Omega, *)$ [3].

Note also that each word from Ω allows for the only expansion into the product of elements from A. Let us remind that the semigroup Ω^* is called the non-empty set Ω, which together with the binary operation $*$ satisfies the associative law $x*(y*z) = (x*y)*z$ for any values of $x, y, z \in \Omega$.

Let us introduce the binary operation ($\circ$) for any $t_\alpha \in E_k^R$ and $t_\beta \in E_p^R$ [10] in the set E^R.

$$t_\alpha \circ t_\beta = t_\alpha + \xi^k t_\beta. \quad (4)$$

It is not difficult to establish by direct calculation (2) that $t_\alpha \circ t_\beta = t_{\alpha\beta} \in E_{k+p}^R$.

For $t_\beta \circ t_\alpha = t_\beta + \xi^p t_\alpha = t_{\beta\alpha} \in E_{k+p}^R$. In this way, $t_{\alpha\beta} \neq t_{\beta\alpha}$ according to its construction so that this binary operation is not commutative, but rather associative. The associativity is verified via immediate calculation.

Thus, the set E^R carrying the introduced binary operation (4), forms the semigroup $(E^R, \circ)$.

Theorem. If Ω^* is the free semi-group, while $(E^R, \circ)$ is the semi-group of right-hand Cantor set first kind points, then the isomorphism exists $t : \Omega^* \to (E^R, \circ)$.

This theorem is rather simple to prove: $t(\alpha * \beta) = t(\gamma)$, since $\alpha * \beta = \gamma$, $\alpha \in \Omega_k$, $\beta \in \Omega_n$, $\gamma \in \Omega_{n+k}$. But on the other hand, $t_\alpha \circ t_\beta = t_\alpha + \xi^k t_\beta = t_\gamma$, where $t_\alpha \in E_k^R$, $t_\beta \in E_n^R$, $t_\gamma \in E_{n+k}^R$. But inasmuch as there exists the one-to-one correspondence of the sets Ω_{n+k} and E_{n+k}^R, then $t(\gamma) = t_\gamma$.

4. Cantor Distribution

Insertion of generalized functions (distributions) concerns the problem of extended functional representations, such as caused, for instance, by the needs of theoretical physics. With this in mind, the function can be considered rather as a mathematical object, exercising influence on another test function, not as a set of values from different points. From this standpoint, the most interesting approach is the one associated with such generalized functions that are produced by measures. As indicated in the Introduction, the construction of theory of the fractal medium electrodynamics should be based on fractal, better to say, Cantor measures. This paper considers employment of such a measure, using the Cantor function α, that is also called the "devil's staircase". The Cantor function is a limited (from above and below) monotonous non-decreasing function, which is the limit of pre-Cantor α_n sequence, i.e. $\alpha(x) = \lim_{n\to\infty} \alpha_n(x)$. All the totality of the pre-Cantor functions $\{\alpha_n\}$ is built on the length $[0,1]$, using the totality of the sets $\{E_n\}$. Fig. 2 shows the function $\alpha_n(x)$ for $n = 2$. As one can gather from Fig. 2, this function $\alpha_2(x)$ has constant values in the rejected open intervals, growing linearly in the closed intervals that belong in this case to E_2. With increasing n, the tangent of the angle of inclination of these straight lines, being equal to $(2\xi)^{-n}$ (or derivative of $\alpha_n(x)$ on E_n), increases, and within the limits of $n \to \infty$ it strives to infinity. Thus, the Cantor function α is a function of jumps in the Cantor set points. In this paper, all derivations are made, using the triad Cantor set with the relation ξ. Whenever it causes no doubt, the parameter ξ is dropped. For example, $\alpha_n(x;\xi) \equiv \alpha_n(x)$.

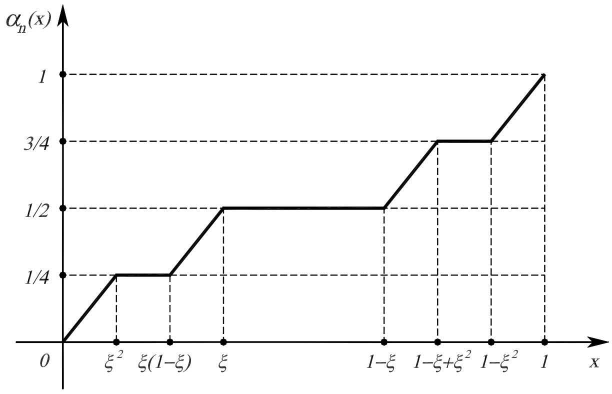

Fig. 2. Cantor function for $n = 2$.

In each of the pre-Cantor sets E_n, one can assign measures that are quite additive functions and, which is especially important, produced by the pre-Cantor functions $\alpha_n(x)$.

$$\mu(E_n) = \int\limits_{E_n} d\alpha_n. \quad (5)$$

The totality of measures $\{\mu(E_n)\}$ forms the sequence that converges to the measure, assigned on the Cantor set, i.e. $\lim_{n\to\infty} \mu(E_n) = \mu(E)$.

The measure $\mu(E_n)$ on the pre-Cantor set E_n determines the generalized function $\triangle_n$ (for $n = 0, 1, 2, \ldots$) according to the formula

$$(\triangle_n, \varphi) = \int\limits_{E_n} \varphi d\alpha_n. \quad (6)$$

We shall use as earlier the abbreviated notation $\triangle_n(x;\xi) \equiv \triangle_n(x)$.

The integral (6) should be interpreted as the Lebesgue-Stieltjes integral [2]. Note also that for $n = 0$ the integral (6) becomes the Rieman integral. In this way, the pre-Cantor functions $\alpha_n(x)$ are generating functions for the measures $\mu(E_n)$.

The generalized function $\triangle_n$ should be regarded as a functional, operating in the space of the test functions $D(Q)$: $\varphi \in D(Q)$. All finite functions, infinitely differentiated in Q, can be referred to this space; $D(Q) = C_0^\infty$. In our case, $Q = R$ is the real axis. The space convergence of the test functions $D(Q)$ $\varphi_k \to \varphi$ is determined by the convergence of the function sequence $\{\varphi_k\}$ and all its derivatives $\varphi_k^{(n)} \to \varphi^{(n)}$. For the generalized functions $\triangle_n$ the functional $(\triangle_n, \varphi)$ assumes the form of

$$\int\limits_{E_n} \varphi d\alpha_n = \int\limits_{R} \triangle_n \varphi dx \equiv (\triangle_n, \varphi). \qquad (7)$$

In this way, for the pre-Cantor generalized functions $\triangle_n$ there is their explicit expression in terms of the derivative from $\alpha_n(x)$

$$\frac{d}{dx}\alpha_n(x) = \begin{cases} 0, & x \notin E_n, \\ \dfrac{1}{(2\xi)^n}, & x \in E_n. \end{cases} \qquad (8)$$

From the expressions (7) and (8), it follows that

$$(\triangle_n, \varphi) = \int\limits_{E_n \subset R} \frac{1}{(2\xi)^n} \chi_{E_n}(x)\varphi(x)dx, \qquad (9)$$

χ_{E_n} is the characteristic function of the set E_n

$$\chi_{E_n} = \begin{cases} 1, & x \in E_n \\ 0, & x \notin E_n. \end{cases} \qquad (10)$$

The expression (9) is not difficult to reduce to the form

$$(\triangle_n, \varphi) = \frac{1}{2^n} \int\limits_0^1 \sum_{\alpha \in \Omega_n} \varphi(\xi^n x + x_\alpha)dx, \qquad (11)$$

where $x_\alpha \in E_n^R$. Note that $(\triangle_n, 1) = 1$ for all n.

The functional (11) shows the impact of the generalized functions $\triangle_n$ on the test function $\varphi(x)$ from the main function space. The result of this impact comes down, actually, to averaging of $\varphi(x)$ on E_n. The supports of the distributions $\{\triangle_n\}$ are the pre-Cantor sets $\{E_n\}$, i.e. $supp\triangle_n = E_n$. Thus, the generalized functions $\{\triangle_n\}$ are the functionals on $D(R)$ and $\triangle_n \in D'(R)$ for all n. $D(R')$ is the linear space of all generalized functions that is dual space to $D(R)$. In the meantime, the convergence in $D'(R)$ is introduced as a weak convergence of the functionals. In this way, the sequence $\triangle_n$ determines the Cantor distribution:

$$\lim_{n\to\infty} (\triangle_n, \varphi) = (\triangle_\xi, \varphi). \qquad (12)$$

Using (11) and (12), we shall represent the Cantor distribution as:

$$(\triangle_\xi, \varphi) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{\alpha \in \Omega_n} \varphi(x_\alpha), \qquad (13)$$
$$(\triangle_\xi, 1) = 1.$$

It is understood that $supp\triangle_\xi = E$.

By resorting to the generalized function insertion procedure, using the measures (5), one can represent $\triangle_\xi$ as an expression, which is most suitable to make calculations with and from which the explicit connection transpires between the Cantor function α, giving birth to the measure $\mu(E)$, and such a generalized Cantor function $\triangle_\xi$ that is the functional:

$$(\triangle_\xi, \varphi) = \int\limits_E \varphi d\alpha. \qquad (14)$$

From (13), it is easy to show that the generalized function $\triangle_\xi$ is a singular generalized function. By comparing the Dirac generalized function δ and $\triangle_\xi$, it must be noted that they are both singular generalized functions and their carriers are the zero measure sets $supp\delta = \{0\}$ and $supp\triangle_\xi = E$.

The generalized Cantor function $\triangle_\xi$ determines the Cantor structuring of the point. From my standpoint, its applicability is quite possible during construction of physical theories only not on smooth manifolds, but also on such manifolds, each point of which is the Cantor set, for example, on the fractal space-time.

5. Semigroup of Operators over the Cantor measure $\mu(E_n)$

Let us bring under consideration the set of operators $\{A_n\}$ that act on the measure set $\{\mu(E_n)\}$, $n = 0, 1, 2, \ldots$ according to the following law:

$$A_k \mu(E_n) = \mu(E_{k+n}). \qquad (15)$$

The operators $\{A_k\}$ assigned by the action (15), form the semigroup G, since it easy to see that $A_k A_n = A_{k+n}$ and the law of associativity is met $A_k(A_n A_j) = (A_k A_n)A_j$. The operator A_0 that leaves the measure unchanged is the unit of this semi-group. The action of the operator $\{A_n\}$ can be expanded on the functional spaces, for example, on the test function space $D(Q)$. In order to obtain the explicit operator type $\{A_k\}$, we assume that any $A_k \in G$ has the following property:

$$(A_k \triangle_n, \varphi) = (\triangle_n, A_k \varphi). \qquad (16)$$

This property of the operators $\{A_k\}$ is chosen in similarity to the differentiation operator action on the generalized functions from $D'(Q)$: $(D^n f, \varphi) = (-1)^n (f, D^n \varphi)$, where n is the order of the derivative. The Fourier transform (29) operator acts in much the same way. In order to obtain the explicit form of the operators $\{A_k\}$, let us assume that in the equation (16) $n = 0$. In this case, $(A_k\triangle_0, \varphi) = (\triangle_0, A_k\varphi)$. Another notation of this expression has the following form:

$$\int_{E_k} \varphi d\alpha_k = \int_R A_k\varphi dx. \quad (17)$$

Employing the expressions (7), (11) и (17), one can obtain the explicit form of the operators $\{A_k\}$, $k = 0, 1, 2, \ldots$

$$A_k\varphi(t) = \frac{1}{2^k} \sum_{\alpha\in\Omega_k} \varphi(\xi^k t + t_\alpha). \quad (18)$$

For $k = 0$, $A_0\varphi = \varphi$. The operators A_k depict the main function space onto themselves. They act to transform $t \to t' = \xi^k t + t_\alpha$ ($t \in R$, $t' \in R$, $t_\alpha \in E_k^R$) and perform the averaging over the right-hand Cantor set first kind points.

The operators $\{A_k\} \in G$ retain the semi-group structure, since it easy to demonstrate via immediate calculations, using (18), that $A_kA_n\varphi = A_{k+n}\varphi$. Really, using the fact that $t_\beta + \xi^k t_\alpha = t_\gamma$, where $\alpha \in \Omega_n$, $\beta \in \Omega_k$, $\gamma \in \Omega_{n+k}$, we obtain:

$$A_nA_k\varphi(t) = \frac{1}{2^n2^k} \sum_{\alpha\in\Omega_n} \sum_{\beta\in\Omega_k} \varphi(\xi^{n+k}t+\xi^k t_\alpha+t_\beta) = \frac{1}{2^{n+k}} \sum_{\gamma\in\Omega_{n+k}} \varphi(\xi^{n+k}t + t_\gamma). \quad (19)$$

From the expressions (15) and (16) at $n \to \infty$, one very important property of invariance follows that is indispensable for calculating the functionals $\triangle_\xi$, precisely:

$$\begin{aligned} &A_k\mu(E) = A_n\mu(E), \\ &(\triangle_\xi, A_k\varphi) = (\triangle_\xi, A_n\varphi). \end{aligned} \quad (20)$$

The equality (20) is just for any values of $n, k = 0, 1, 2, \ldots$.

To supply an example, one can calculate the concrete functional [10]:

$$I(\nu) = (\triangle_\xi, x^\nu) = \int_0^1 x^\nu d\alpha. \quad (21)$$

By employing the relations (20) for A_1 and expression (18), it is easy to obtain the value of the functional $I(\nu)$:

$$I(\nu) = \int_0^1 A_1x^\nu d\alpha = \int_0^1 \frac{(\xi x)^\nu + (1 - \xi + \xi x)^\nu}{2} d\alpha. \quad (22)$$

In this way, one can obtain the functional relationship for the functionals $I(\nu)$:

$$I(\nu) = \frac{(1 - \xi)^\nu}{2 - \xi^\nu} \sum_{k=0}^{\infty} \binom{\nu}{k} (\xi^{-1} - 1)^{-k} I(k), \quad (23)$$

$$\binom{\nu}{k} = \frac{\nu(\nu - 1)\ldots(\nu - k + 1)}{k!}.$$

For $\nu = m$, $m \in Z$, and using the fact that $I(0) = 1$, the expression (23) acquires the form:

$$I(m) = \frac{(1 - \xi)^m}{2(1 - \xi^m)} \times \sum_{k=0}^{m-1} \binom{m}{k} (\xi^{-1} - 1)^{-k} I(k). \quad (24)$$

For $m = 1$, $I(1) = \frac{1}{2}$. In this case, the Lebesgue-Stieltjes integral (21) is not different from the Rieman integral for the function x^m. Although for $m = 2$, $I(2) = \frac{1}{2(1 + \xi)}$. This result is drastically different from the Rieman integral value, which is equal to $\frac{1}{3}$. Both results coincide for $\xi = \frac{1}{2}$. Note also that $\frac{1}{2(1 + \xi)} > \frac{1}{3}$ for $0 < \xi < \frac{1}{2}$.

Below, certain expressions are given for some functionals that have been calculated, using the above procedure:

$$(\triangle_\xi, e^{\eta t}) = e^{\eta/2} \prod_{n=0}^{\infty} \mathrm{ch}\left[\frac{(1 - \xi)\xi^n}{2}\eta\right], \quad (25)$$

$$\begin{aligned} &(\triangle_\xi, e^{i\pi a t}) = e^{i\pi a/2}\gamma_\xi(\pi a), \\ &\gamma_\xi(\pi a) = \prod_{n=0}^{\infty} \cos\left[\frac{(1 - \xi)\xi^k}{2}\pi a\right]. \end{aligned} \quad (26)$$

By comparing the expressions for the functionals (26) and (13), it is easy to derive the expression to be used in future:

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\alpha\in\Omega_n} e^{it_\alpha\omega} = e^{i\omega/2}\gamma_\xi(\omega). \quad (27)$$

In this way, the invariant (20) provides for a mere opportunity of calculating the functionals $\triangle_\xi$.

6. Fourier Transform of Cantor Distribution

Let φ be the locally integrated function to R^n. The Fourier transform of this function is represented as:

$$F[\varphi](\xi) = \frac{1}{(2\pi)^n} \int_{R^n} e^{-i(\xi,x)} dx. \quad (28)$$

In the meantime, it is implied that $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ are the pseudo-Euclidean coordinates in the general case. The scalar product (ξ, x) of these vectors is introduced, as usual, using the pseudo-Euclidean metrics. The integral (28) is considered convergent in the principal value meaning.

In order to construct the generalized function Fourier transform, the space of test functions (fast-decreasing) $\mathcal{L}$ and that of generalized functions (slow-growing) $\mathcal{L}'$ [4] are tabled for consideration.

The space of principal functions $\mathcal{L} = \mathcal{L}(R^n)$ consists of all the functions, infinitely differentiated in R^n, decreasing at $|x| \to \infty$ together with all derivatives faster than any degree $|x|^{-1}$. While $\mathcal{L} \supset D$.

The slow-growth generalized function is any linear continuous functional in the test function space $\mathcal{L}$. The generalized function space $\mathcal{L}' = \mathcal{L}'(R^n)$ consists of the set of all the slow-growth generalized functions. It is evident that $\mathcal{L}' \subset D'$.

Let $f(x) \in \mathcal{L}'$, while $\varphi(x) \in \mathcal{L}$, then, by definition the Fourier transform of the distribution can be represented by the equality:

$$(F[f], \varphi) = (f, F[\varphi]). \tag{29}$$

The operation $\varphi \to F[\varphi]$ being linear and continuous from $\mathcal{L}$ into $\mathcal{L}$, the functional, determined by the right-hand part of the equality (29), represents a generalized function from $\mathcal{L}'$. The operation $f \to F[f]$ is linear and continuous from $\mathcal{L}'$ into $\mathcal{L}'$ [4].

If f is a generalized function with the compact support, then, it is a slow-growth generalized function and its Fourier transform exists [4]. It is exactly this case that is of immediate interest, because the generalized Cantor function has the compact support $supp\triangle_\xi = E$. Note that the generalized functions $\{\triangle_n\}$ too have the compact supports $supp\triangle_n = E_n$, which indicates the existence of the Fourier transform for them, as well. Based on the theorem, given in [4], the Fourier transform $F[f]$ for $f \in \mathcal{L}'$ and, consequently, for $\triangle_\xi \in \mathcal{L}'$ can be represented as:

$$F[\triangle_n](k) = \left(\triangle_n, \eta(x)\frac{e^{ikx}}{2\pi}\right), \tag{30}$$

where $\eta \in D$ being equal to unity in the vicinity of $supp\triangle_n$.

Employing the computational algorithm for the functional (11), it is easy to obtain the Fourier transform for $\triangle_n$:

$$F[\triangle_n] = \frac{1}{2^n}\frac{1}{2\pi}\sum_{\alpha\in\Omega_n} e^{ikx_\alpha} e^{ik\xi^n/2}\frac{\sin(k\xi^n/2)}{k\xi^n/2}. \tag{31}$$

In this way, the Fourier transform of the Cantor distribution has the form:

$$F[\triangle_\xi] = \lim_{n\to\infty} F[\triangle_n] = \lim_{n\to\infty}\frac{1}{2^n}\frac{1}{2\pi}\sum_{\alpha\in\Omega_n} e^{ikx_\alpha}, \tag{32}$$

where $x_\alpha \in E_n^R$.

Using the earlier obtained result (27), $F[\triangle_\xi]$ can be transformed into the following form:

$$F[\triangle_\xi](k) = \frac{e^{ik/2}\gamma_\xi(k)}{2\pi}. \tag{33}$$

Let us consider in $\mathcal{L}'$ the Fourier inversion, indicated in terms of F^{-1} according to the formula:

$$F^{-1}[f] = F[f(-x)](2\pi)^n, \tag{34}$$

where $f(-x)$ is the reflection of $f(x)$. The operation F^{-1} is linear and continuous from $\mathcal{L}'$ into $\mathcal{L}'$. The operation F^{-1} is inverse to the operation F, i.e.

$$F^{-1}[F[f]] = f, \qquad F[F^{-1}[f]] = f, \qquad f \in \mathcal{L}'. \tag{35}$$

Thus, the Cantor distribution can be expressed in terms of the Fourier inversion:

$$F^{-1}[e^{ik/2}\gamma_\xi(k)] = 2\pi\triangle_\xi. \tag{36}$$

On the other hand, it is easy to demonstrate that

$$F\left[\triangle_\xi\left(x + \frac{1}{2}\right)\right](k) = \frac{1}{2\pi}\gamma_\xi(k). \tag{37}$$

Fig. 3 shows construction of the function $\gamma_\xi(x)$ for various values of ξ.

7. Transition Scattering of Fractal Type

The charge-dissipated permittivity waves are known to produce electromagnetic (transition) radiation. This process is known as transition scattering due to two reasons (at least): firstly, this phenomenon is quite feasible on the charge at rest, secondly, this type radiation is emitted all over the space continuously, not in a region, localized near one boundary. Owing to all of the above, interferential phenomena of the electromagnetic waves occur [6,7]. Transition scattering has been studied well enough during permittivity variations ε according to the periodic law. Permittivity can depend only on concentration of the medium N. If a longitudinal acoustic wave propagates through such a medium, then the density $N = N_0 + \triangle N\cos(\vec{k}\vec{r} - \omega_0 t)$. Thus,

$$\varepsilon = \varepsilon_0 + \varepsilon_1\cos(\vec{k}\vec{r} - \omega_0 t), \tag{38}$$

ε_1 is a variation of ε, as caused by the changing $N(\varepsilon_1 \sim \triangle N)$.

The inductance $\vec{D}$ associated with the charge electrostatic field appears around the charge, placed in the medium with the permittivity ε_0:

$$\vec{D}(\vec{r}) = \varepsilon_0\vec{E}_0(\vec{r}), \qquad \vec{E}_0(r) = \frac{q\vec{r}}{\varepsilon_0 r^3}. \tag{39}$$

With appearance of the permittivity wave $\vec{D}(\vec{r}) \to \vec{D}(\vec{r}, t) = \varepsilon(\vec{r}, t)\vec{E}(r, t)$. In the first approximation

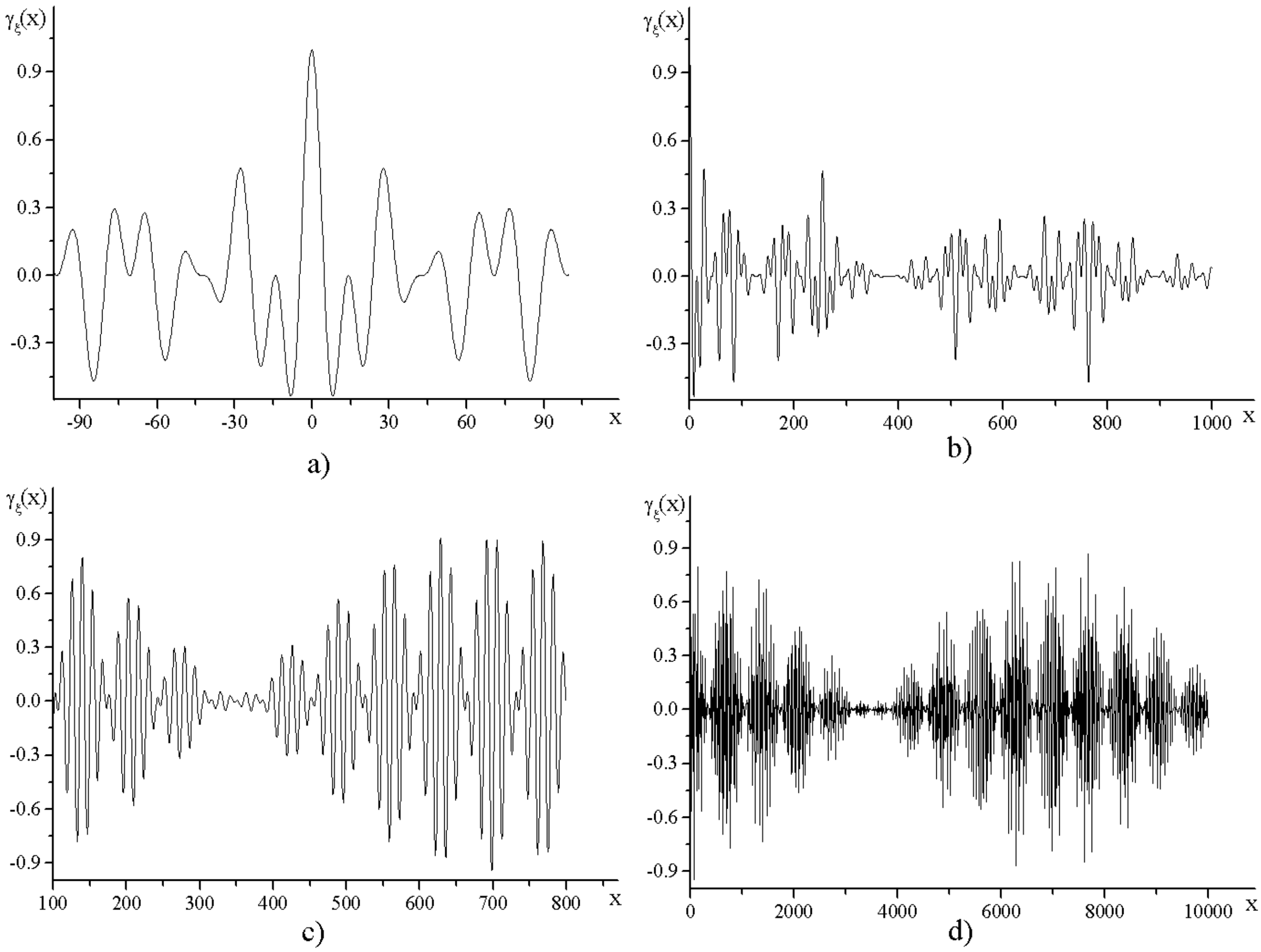

Fig. 3. Charts of the function $\gamma_\xi(x)$ for various ξ: a) $\xi = \frac{1}{3}$, b) $\xi = \frac{1}{3}$, c) $\xi = \frac{1}{10}$, d) $\xi = \frac{1}{10}$.

$|\varepsilon_1| \ll |\varepsilon_0|$ a variable polarization [7] emerges around the charge:

$$\delta\vec{P} = \frac{\delta\vec{D}}{4\pi} = \frac{\varepsilon_1\vec{E}_0}{4\pi}\cos(\vec{k}\vec{r} - \omega_0 t), \qquad (40)$$

Such variable polarization brings about the appearance of an electromagnetic wave with the frequency ω_0 that spreads out from the charge.

This paper studies such transition scattering that is caused by the non-periodic dissipative permittivity perturbation:

$$\varepsilon(\vec{r}, t) = \varepsilon_0 + \varepsilon_1\gamma_\xi(\vec{k}\vec{r} - \omega_0 t). \qquad (41)$$

The permittivity burst (41) (medium density perturbation) may occur due to different causes, for example, shock waves in this medium or concentration inhomogeneities in a plasma. This paper models the transition scattering process, using the function $\gamma_\xi(x)$ whose characteristic feature is connection of its Fourier transform with the Cantor set. The behavioral characteristics of this function for various values of ξ are given in Fig. 3.

The Fourier transform $\vec{D}(\vec{k}, \omega)$ for the inductance $\vec{D}(\vec{r}, t)$ has the form (28):

$$\vec{D}(\vec{k}, \omega) = \frac{1}{(2\pi)^4}\times \int d^3\vec{r}dte^{-i(\vec{k}\vec{r}-\omega t)}\varepsilon(\vec{r}, t)\vec{E}(\vec{r}, t). \qquad (42)$$

In this expression, by substituting the value $\varepsilon(\vec{r}, t)$ from (41) and using the formula (36), it is easy to obtain the expression for $\vec{D}(\vec{k}, \omega)$:

$$\vec{D}(\vec{k}, \omega) = \varepsilon_0\vec{E}(\vec{k}, \omega) + 2\pi\varepsilon_1\int\limits_{-1/2}^{1/2} d\nu\vec{E}(\vec{k} - \nu\vec{k}_0, \omega - \nu\omega_0)\triangle_\xi\left(\nu + \frac{1}{2}\right). \qquad (43)$$

It is well-known how the Fourier components of inductance of the field $\vec{D}(\vec{k}, \omega)$ are connected with components of the Fourier polarization vector $\delta\vec{P}(\vec{k}, \omega)$:

$$\vec{D}(\vec{k}, \omega) = \varepsilon_0\vec{E}(\vec{k}, \omega) + 4\pi\delta\vec{P}(\vec{k}, \omega). \qquad (44)$$

In this way,

$$\delta\vec{P}(\vec{k},\omega)=\frac{\varepsilon_1}{2}\int\limits_{-1/2}^{1/2}dx\triangle_\xi\left(x+\frac{1}{2}\right)\times \vec{E}(\vec{k}-x\vec{k}_0,\omega-x\omega_0). \quad (45)$$

Let us consider transition scattering in the case of the charge at rest. The field equation in this case is written down in the following way:

$$rotrot\vec{E}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{D}=0. \quad (46)$$

From these equations one can easily obtain the Fourier transform in the case under consideration (43)

$$(k^2\delta_{ij}-k_ik_j-\frac{\omega^2}{c^2}\varepsilon_0\delta_{ij})E_j(\vec{k},\omega)=\frac{2\pi\omega^2}{c^2}\varepsilon_1\times \int\limits_{-1/2}^{1/2}d\nu E_i(\vec{k}-\nu\vec{k}_0,\omega-\nu\omega_0)\triangle_\xi\left(\nu+\frac{1}{2}\right) \quad (47)$$

As usual, the summation is made over the reiterative indexs.

Let us consider that $|\varepsilon_1|\ll|\varepsilon_0|$. In this case, the right-hand part of the equality (47) can be regarded as perturbation, associated with changing polarization in the proximity of the charge. Let us use the immovable charge Fourier field components in the first order of the perturbation theory:

$$\vec{E}_{\vec{k},\omega}=-\frac{4\pi iq\vec{k}}{(2\pi)^3\varepsilon_0k^2}\delta(\omega). \quad (48)$$

Thus, the polarization perturbation in the proximity of the charge can be obtained by substituting (48) into (45):

$$\delta\vec{P}(\vec{k},\omega)=\frac{iq\varepsilon_1}{4\pi^2\varepsilon_0\omega_0}\frac{(\vec{k}-\omega\vec{k}_0/\omega_0)}{(\vec{k}-\omega\vec{k}_0/\omega_0)^2}\times \triangle_\xi\left(\frac{\omega}{\omega_0}+\frac{1}{2}\right). \quad (49)$$

If, for the sake of precision, dissipated waves are considered to be transverse (in relation to the vector $\vec{k}$), then the expression (47) shall assume the following form:

$$\left(k^2-\frac{\omega^2}{c^2}\varepsilon_0\right)\vec{E}^\tau(\vec{k},\omega)= \frac{4\pi\omega^2}{c^2}\frac{[\vec{k},[\vec{k},\delta\vec{P}(\vec{k},\omega)]]}{k^2}. \quad (50)$$

The expressions (50) and (49) provide the opportunity for computing the dissipated transverse wave field:

$$\vec{E}^\tau(\vec{k},\omega)= -\frac{iq\omega^3[\vec{k},[\vec{k}_0,\vec{k}]]}{\pi(\omega_0kc)^2(k^2-\omega^2\varepsilon_0/c^2)(\vec{k}-\omega\vec{k}_0/\omega_0)^2}\times \frac{\varepsilon_1}{\varepsilon_0}\triangle_\xi\left(\frac{\omega}{\omega_0}+\frac{1}{2}\right). \quad (51)$$

The expression (51) is similar to the electric field Fourier component one, as obtained in the paper [6] for the permittivity wave type (40). The main difference is in the fact that the formula, employed in (51) for this paper, includes the Cantor distribution $\triangle_\xi$, not δ -function. This difference is indicative of the fact that, in this case, the electromagnetic radiation emission occurs rather across a certain frequency range, such as determined by the Cantor distribution, than at one frequency.

The paper [8] demonstrates that the transverse wave radiation intensity is described by the expression:

$$W\left(\omega,\vec{k}/k\right)d\omega do= (2\pi)^6\frac{\omega^4}{c^3}\sqrt{\varepsilon_0}\left|\delta\vec{P}^\tau(\vec{k},\omega)\right|^2d\omega do, \quad (52)$$

where $\delta\vec{P}^\tau(\vec{k},\omega)$ is the transverse polarization which is found easy from the equation (49):

$$\delta\vec{P}^\tau(\vec{k},\omega)=\frac{[\vec{k},[\vec{k},\delta\vec{P}(\vec{k},\omega)]]}{k^2}, \quad (53)$$

where $\vec{k}=\frac{\vec{k}}{k}\frac{\sqrt{\varepsilon_0}\omega}{c}$.

In this way, the expressions (49), (51) and (52) indicate that during transition fractal scattering the electromagnetic wave spectrum is localized near the frequencies $\omega=\omega_0\left(\omega_\alpha-\frac{1}{2}\right)$. The numbers ω_α belong to the Cantor set points and are determined by the expression (2).

8. Conclusions

This paper is a continuation of the sequel, written on the transition fractal radiation emission theory construction [9], but it is of more general nature and its results can be used during the fractal electrodynamics theory construction.

The Cantor distribution is brought under consideration in this paper. Its derivation is based on the Cantor measures. The algorithm for functional computations is brought up, using the Cantor distribution.

An example is provided for the permittivity (41) dissipative burst-induced transition scattering during its incidence on the charge. Such a type of the permittivity space-time function forms a wide-range fractal radiation emission spectrum during scattering on the charge at rest, with the radiation emission frequencies being localized in the Cantor set points. A rather complicated kind of dielectric burst can be formed by shock waves in the dielectric materials, by turbulent movements in plasma-like media and the fluids.

In this case, transition scattering can serve, on the one hand, as a source of wide-range electromagnetic radiation, on the other hand, it can be used for the diagnostics of random dynamics of various media.

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References

[1] Aleksandroff P.S. Introduction to the Theory of Sets and General Topology. - M.: Nauka. Editors-in-Chief of Phys. - Math. Literature. - 1977. - 368 p.

[2] Kolmogorov A.N., Fomin, S.V. Fundamentals of the Function Theory and Functional Analysis. - M.: Nauka. Editors-in-Chief of Phys. - Math. Literature. - 1968. - 496 p.

[3] Lallement G. Semigroups and Combinatorial Additions. - M.: Mir. - 1985. - 440 p.

[4] Vladimirov V.S. Generalized Functions in Mathematical Physics. - M.: Nauka. Editors-in-Chief of Phys.- Math. Literature. - 1979. - 320 p.

[5] Eric Olivier. Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures // Nonlinearity. - 1999. - V. 12. - P. 1571-1585.

[6] Ginsburg V.L., Tsytovich V.N. On transition Dissipation// ZHETF. - 1973. - V. 65. - P. 1818-1824.

[7] Ginsburg V.L., Tsytovich, V.N. transition Radiation and transition Dissipation. - M.: Nauka. - 1984. - 360 p.

[8] Kolessov V.V. Electromagnetic Radiation Energy of Arbitrary Sources in a Transparent Dispersive Medium. On the Effects of Interference on the Dissipative Processes. // ZHETF. - 1994. - V. 106, N 1(7). - P. 77-89.

[9] Bolotov V.N. Transition Fractal Radiation. // - Technical Physics. - 2000. - V. 45, N 12. - P. 1604-1607.

[10] Gural'nik T.Yu. Private Communication.