Modern Helmholtzian Electrodynamics as a Covering Classical Electromagnetic Theory

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Abstract

The discussion of the historical background of the 19th century electromagnetic theory has shown that from the standpoint of modern scientic method, the Hertz experiments on propagation of electromagnetic interactions cannot be considered as conclusive at many points as it is generally implied. It has been found that alternative Helmholtz’s electrodynamics did not contradict Hertz’s experimental observations. Mathematical analysis of the conventional electromagnetic theory showed that numerous ambiguities are related to the treatment of time behaviours. Those difficulties turn out to be cleared up by distinguishing between implicit and explicit time dependencies. It provides self-consistency for mathematical description of electromagnetic theory by advocating the explicit use of full time derivatives in the mathematical formulation of Maxwell’s equations. This approach covers conventional electromagnetic theory based on partial time derivatives. The covering theory is showed to possess all necessary relativistic invariance properties for inertial frames of references. The idea of non-local interactions is enclosed into the framework of Helmholtzian electromagnetic theory as unambiguous mathematical feature. In this work we make a point that Helmholtz’s foundations and modern Helmholtz-type electrodynamics recently developed by the authors and reviewed here promise, in general, an altogether more logical solution to self-consistent classical electrodynamics and its reconciliation with quantum mechanics.
1. Introduction

There is no need to argue that classical electrodynamics is one of the corner-stones of modern physics. At first stages the development of electromagnetic theory proceeded in accordance with Newtonian traditional outlook on the world based on the instantaneous action-at-a-distance (IAAAD). Faraday’s discovery of induction highlighted limited validity of IAAAD in describing electromagnetic phenomena. A notion of local (contact) field was proposed by Faraday not to incorporate but to replace Newtonian IAAAD. As a result, the state of electrodynamics in the middle of the 19th century was characterised by opposition of a few alternatives. Hertz’s discovery of electromagnetic waves played a crucial role in choosing a definitive modern version of classical electrodynamics. However, nowadays only a few researchers are aware of the fact that Helmholtz’s electrodynamics was also consistent with Hertz’s experiments as well as with all known 19th century’s experimental data (see, for instance, [1]). Why and how Maxwell’s electrodynamics became a favourite needs to be revisited in detail.

2. Historical background of classical electrodynamics

In order to appreciate the difficulty and the importance of the task undertaken by Hertz in his experimental investigations, it is worth recalling the uncertain and highly controversial state of electrodynamics at that time. Hertz himself was trained in the research tradition of the Berlin school headed by Helmholtz who from the middle 1860’s, had sought to clarify existing principles in electromagnetic theory and to reach a consensus between the two major directions in electromagnetic research of that time, namely, Newton’s instantaneous action-at-a-distance concept as used by Weber, and Faraday’s contact action concept as developed by Maxwell. By the time of Helmholtz’s first attempt at reconciliation (1870), the theoretical schemes of Weber and Maxwell had successfully incorporated all previously well-established descriptions and empirical facts, such as the electric potential theory (electrostatics), Ampere’s magnetostatics and Faraday’s theory of induction.

Weber developed his theory (1848) in accordance with the Newtonian program, which prescribed that all forces between pairs of particles should be radial, acting directly through space (i.e. along the line between the particles) without any observable material mediator. Restriction on this radial description of electromagnetic forces came from Ampere, who understood that instantaneous means no delay, hence no aberration. Any aberration attending the finite propagational velocity of interactions would imply non-radial forces. However, the existence of radial forces as a basic assumption of instantaneous action-at-a-distance (IAAAD) theories was confirmed by Ampere experimentally to the degree of accuracy available at that time. Thus, electric and magnetic interactions were thought to be completely analogous to gravitational attractions which, according to astronomical observations, had no detectable aberration and always acted along the line joining the simultaneous positions of two bodies. Bearing in mind the accuracy-limit of astronomical data at that time, Laplace (1799) [2] calculated, measuring possible aberration effects, that the speed of propagation of gravitational interaction had to exceed at least eight orders of magnitude the speed of light. (Recently, Van Flandern claimed to rise that limit two orders of magnitude [3]: "...Perhaps contact binary star systems place the tightest constraints on a lower limit to the speed of propagation of gravity. Unless the speed of gravity exceeds $10^{10}$ times the speed of light, such systems would fly apart within a few hundred years...").

Laplace’s respectable conclusion gave major support to the validity of the IAAAD concept, leaving open the question of the physical cause of gravity. On this latter subject, a conceptual arm-wrestle was initiated between supporters and detractors of IAAAD. Newton himself had already thought that some physical mediator must exist [4]. "It is absurd," he said, "to suppose that gravity is innate and acts without a medium, either material or immaterial". Many eminent scientists like Laplace explained the phenomenon as due to an ‘impulsion’ of some immaterial fluid, but did not find it appropriate at the time to search for a reliable physical explanation of the cause of interaction, and so they bequeathed this task to future generations. "There is no need at all," declared Laplace (1796) [5], "to posit vague causes, impossible to submit to analysis, and which the imagination modifies to its liking in order to explain these phenomena".

At that stage, it is not surprising that for many of the adversaries of IAAAD these suggestions of ‘immateriality’ could not be dissociated from spiritual, religious and other non-scientific notions. This path of reasoning led them to conclude that if the IAAAD concept did not imply any material mediator, then it did not imply any physical mediator at all (compare it, for instance, with nowadays explanation of interaction mediated by material as well as by non-material virtual particles). Thus, in their opinion IAAAD became inconceivable from the point of view of common-sense logic. This prepared the ground for the reappearance of an alternative Aristotelian concept, namely, that of action by local contact (or contact action). Thought-provoking examination of that resurgence and related modern topics on the debate between far- and local actions can be found in a well-written book by Graneau’s [6].
In its modern form, the concept of local (contact) field was reintroduced by Faraday and it appeared to give a more realistic physical description of the phenomena, based on the causality of local interactions. This way of reasoning attracted Maxwell, who tried to work out his own comprehensive field theory based on Faraday’s concept. However, Maxwell himself was aware of the provisional, ‘scaffolding’ status of his rationale. Moreover, he encountered some conceptual difficulties, since he had incorporated all the basic IAAAD results such as electrostatics and magnetostatics without any modification. With regard to the lines of force treated by Faraday as the representation of a material field, Maxwell’s own position was still undefined, but he cautiously dealt with them as if they were lines of flow of an incompressible, imaginary fluid. As Maxwell stated [7]: "The substance here treated of must not be assumed to possess any of the properties of ordinary fluids except those of freedom of motion and resistance to compression. It is not even a hypothetical fluid which is introduced to explain actual phenomena. It is merely a collection of imaginary properties which may be employed for establishing certain theorems in pure mathematics in a way more intelligible to many minds and more applicable to physical problems than that in which algebraic symbols alone are used".

Consequently, being in a static limit, mathematically equivalent to older IAAAD theories, the status of contact field theory could not have been considered free of ambiguity [8]. In other words, in this static limit, electric and magnetic fields behaved very much as flows of an ideal, incompressible fluid, in which case they were indistinguishable from IAAAD. These uncertainties in Maxwell’s original theoretical scheme were later summarised clearly and concisely by Hertz, who wrote [9]: "...Maxwell’s own representation does not indicate the highest attainable goal; it frequently wavers between the conceptions which Maxwell found in existence, and those at which he arrived. Maxwell starts with the assumption of direct actions-at-a-distance; he investigates the laws according to which hypothetical polarisations of the dielectric ether vary under the influence of such distance-forces; and he ends by asserting that these polarisations do really vary in this way, but without being actually caused to do so by distance-forces. This procedure leaves behind it the unsatisfactory feeling that there must be something wrong about either the final result or the way which led to it."

As an approach to clarifying these uncertainties, let us examine the essential distinction between the conceptual foundations of IAAAD and those of contact-field doctrines. At the beginning of the nineteenth century, the possibility of a final explanation of IAAAD as the basis of gravitational and electric forces had not been completely ruled out. Moreover, due to the rapid development of theoretical hydrodynamics, the attitude towards IAAAD changed from the summary rejection of its unphysical status to an awareness of a deep similarity between the potential function and the velocity-field of a fluid. It had been realised that the main difference between IAAAD and Faraday’s ‘field’ was the fact that in IAAAD a potential need not necessarily describe a material property of anything, whereas for Faraday it was the property of a material substance which could be observed in ways familiar to ordinary matter such as, for instance, liquids and gases. Like any other material substance, therefore, Faraday’s ‘field’ could be regarded as something movable with positive kinetic energy and, therefore, detectable empirically. This suggests that an important criterion, as Hesse puts it [10]: "...in deciding whether or not a field is to be regarded as a physically continuous medium rather than a mere mathematical device lies in its possession of detectable properties other than the one property for which it was introduced. A condition of this kind is often suggested as a of the physical ‘reality’ of a theoretical entity, and it led Faraday to express his dissatisfaction with Newtonian gravitation. But independent detection was not the only consideration which weighed with the nineteenth-century physicists. They were prepared to regard a field as a physically continuous medium on other and less stringent terms, for example, if propagation was affected by material changes in the intervening space, if it took time, if a mechanical model could be imagined for the action of a medium in producing the observed effect, or if energy could be located in the space between interacting bodies. Any of these three conditions might be regarded as sufficient and no one of them was individually necessary. Thus, gravitation remained an action at a distance throughout the nineteenth century, in spite of its description by a potential theory, because it did not satisfy any of these criteria, whereas the electromagnetic field theory began to take on the characteristics of continuous action because it satisfied all of them. It is sometimes suggested in modern (post-relativity) works that a finite velocity of propagation is a necessary as well as sufficient condition for continuous action, and that this is why instantaneous gravitation could not be regarded as such an action, but in classical physics there is instantaneous transmission of pressure and of longitudinal waves in an incompressible medium, and this would certainly be regarded as continuous action."

From this survey of the two opposite conceptual foundations, we may conclude that the nineteenth-century physicists considered as sufficient conditions for validity of Faraday-Maxwell’s field concept:

1. propagation of fields producing material changes in the surrounding space;
2. time-delay in propagation;
3. experimental observation of the energy located between interacting bodies.

Nevertheless, as will be discussed later, the above-mentioned conditions should be considered only as necessary, not sufficient for establishing the existence of Faraday-type contact fields, since there might be a third alternative which would combine both IAAAD and Faraday’s contact-field features in a single scheme. In fact, the examination of the historical background of the nineteenth century electrodynamics, such a third alternative did actually exist and this was the so-called compromise theory of Helmholtz.

By the time Helmholtz became actively involved in resolving problems of electromagnetism, in the middle of the 1860’s, Weber’s and Maxwell’s supporters had already been locked in a lengthy and futile conflict. Helmholtz attempted to make a decisive choice between them by constructing his own mathematical scheme and designing crucial experiments to weigh in favour of either Weber’s theory or Maxwell’s. Being aware of their respective advantages and disadvantages, Helmholtz [11] attempted to elaborate his compromise approach aimed at combining the important elements of the two theories. However, he did not accept the idea of Maxwell’s light-ether and adopted, instead, the concept of the dielectric and diamagnetic medium. The condition of infinite polarizability, in this ether model, required the charge to behave like an incompressible fluid, making all circuits closed as in Maxwell’s theory. When trying to arrive at results similar to Maxwell’s without losing the elements of action-at-a-distance, Helmholtz assumed that the electrostatic forces are constantly present as a field in space and that the change in the polarisation or the displacement of the charges signalled the change in the electrostatic field. As discussed in [12, 13], under these assumptions, Helmholtz successfully derived generalised equations very similar to those of Maxwell and found that in a limited case they yield equations identical to Maxwell’s. Solving these equations for a homogeneous dielectric medium, he arrived at the wave equations for electric and magnetic polarisations, respectively, with undetermined constant k (see, for instance, [12, 13], or [14]). Equations for electric polarisation and magnetic polarisation have solutions for transverse waves whereas equation for electric polarisation also defines longitudinal waves.

The conciliatory aspect of Helmholtz’s approach resulted in the following peculiarity. The results of Maxwell’s theory can be obtained by setting k = 0 (and, of course, by setting the values of electromagnetic constants in correspondence with the model of the light-ether). In addition to the ordinary transverse electromagnetic waves already confirmed by Maxwell, Helmholtz discovered the existence of longitudinal electric waves which turned out to be instantaneous at the Maxwellian limit (k = 0). Interpretation of this conclusion and its consequences became a hard nut to crack for all contemporary electrodynamicists. Maxwellian followers (Heaviside, FitzGerald, Lodge etc.) refused to accept Helmholtz’s theory because they found his conceptions entirely foreign to Maxwell’s view of the transmission of interaction but ignored the fact that both theories could be considered mathematically equivalent. (A comprehensive discussion on this mathematical equivalence was made by O’Rahilly [15]).

Helmholtz’s attempt at a more consistent reformulation of the contemporary electrodynamics theories could not, however, resolve the problem of which approach to favour. Therefore, the need for decisive and reliable experimental data was urgent when Hertz became interested in electromagnetic research.

3. Conventional and alternative interpretations of Hertz’s crucial experiments

In 1879 Helmholtz proposed a prize competition, "To establish experimentally a relation between electromagnetic action and the polarisation of dielectrics" and urged his student Hertz to take up the challenge. At first, Hertz declined, discouraged by the poor prospects of success at that time. Later on, he began investigating the problem for his own interest. In 1886–88, at Karlsruhe, he attempted to establish the compatibility of the theories of Helmholtz and Maxwell in a new series of experiments. He designed his measurement-procedure, taking into account Helmholtz’s separation of the total electric force into the electrostatic and electrodelectric parts to which different velocities of propagation were ascribed. In Hertz’s words [16]: "The total force may be split up into the electrostatic part and electrodynamic part; there is no doubt that at short distances the former, at great distances the latter, preponderates and settles the direction of the total force".

He was aware of the different rate of decrease with distance for each of these forces. According to Coulomb’s law, the electrostatic component was thought to be proportional to the inverse square of the distance, whereas the electrodelectric part was only proportional to the inverse of the distance (in the conventional theory of the Lienard-Wiechert potential it would correspond to decreasing rates of the bound-field, or longitudinal, component and the radiation field, or transverse component, respectively). It is not surprising, therefore, that the systematic identification of the components of the total force appeared to be an extremely difficult task for any reliable experiment of that time because the
magnitude and the phase of the total force as a superposition of components propagating with different speeds and decreasing rates would be constantly changing with distance. Even for modern experimental techniques this task is not an easy one because in addition to the knowledge of the total force, one of the components should also be verified quantitatively.

Hertz decided to carry out experiments. Despite the uncertainty of some of the results of his first experiments, Hertz [17] could already make an important, albeit only qualitative, conclusion: "There are already many reasons for believing that the transversal waves of light are electromagnetic waves; a firm foundation for this hypothesis is furnished by showing the actual existence in free space of electromagnetic transversal waves which propagated with a velocity akin to that of light". This qualitative result was published in a paper entitled "On the Finite Velocity of Propagation of Electromagnetic Action" (1888) which nowadays, according to the established historiography of physics, is considered a classical reference in which the difficult task of proving the finite propagation velocity of electromagnetic interactions in air has been achieved.

Nevertheless, it should be noted that the title of Hertz's paper is perhaps misleading nowadays, because conventional Maxwellian electrodynamics does not employ the Helmholtzian 'action' terminology, nor does it split up the total electric force into electromagnetic and electrostatic parts. However, for Hertz's contemporaries who supported the Helmholtz theory, the underlying meaning of the presented results was clear enough: Hertz's experiment could qualitatively conclude about the finite propagation of the electromagnetic (transverse) part, but could say nothing definite about the electrostatic (longitudinal) component. Looking carefully through the same paper, we find Hertz declaring [18]: "From this it follows that the absolute value of the first of these is of the same order as the velocity of light. Nothing can as yet be decided as to the propagation of electrostatic interactions".

Moreover, some of Hertz's measurements [19] tended towards the instantaneous nature of the electrostatic mode, but he was still not convinced of this instantaneity and preferred to be cautious, since his method was unable to provide him with any reliable quantitative results: "Since the interferences undoubtedly change sign after 2.8 meters in the neighbourhood of the primary oscillation, we might conclude that the electrostatic force which here predominates is propagated with infinite velocity. But this conclusion would in the main depend upon a single change of phase... If the absolute velocity of the electrostatic force remains for the present unknown, there may yet be adduced definite reasons for believing that the electrostatic and electromagnetic forces possess different velocities".

It should be especially stressed that Hertz at this stage was fully aware of the need for additional experiments to cast some light of certainty on the electrostatic part [18]: "It is certainly remarkable that the proof of a finite rate of propagation should have been first brought forward in the case of a force which diminishes in inverse proportion to the distance [electrodynamic part], and not to the square of the distance [electrostatic part]. But it is worth while pointing out that this proof must also affect such forces as are inversely proportional to the square of the distance".

It is interesting briefly to follow Hertz's gradual shift towards accepting Maxwell's field concepts. He started with Helmholtz's theory, and his conversion to Maxwell's viewpoint was an uneasy process, possibly never fully completed due to his (Hertz's) premature death. He began analysing the underlying concepts in the Maxwellian limit of Helmholtz's theory but his final interpretation became essentially different in form from what had been commonly accepted by Helmholtz and his supporters. More specifically, Hertz uncritically assumed that in Maxwell's limit the instantaneous longitudinal component should be excluded from consideration in Helmholtz's original theory. All forces then became explicitly time dependent. This was a drastic departure from his mentor's philosophical foundations because Helmholtz rejected time dependent forces (he admitted only implicit time dependence upon space position) and was deeply convinced that "...nature could only be comprehended through invariant causes"... Helmholtz viewed electromagnetic interactions — indeed, all interactions, — as instantaneous and bipartite..." [20] and, therefore, could attribute interactions only to longitudinal components (electrostatic forces). In Hertz's own words [21]: "...Helmholtz distinguishes between two forms of electrical force the electromagnetic and the electrostatic to which, until the contrary is proved by experience, two different velocities are attributed. An interpretation of the experiments from this point of view could certainly not be incorrect, but it might perhaps be unnecessarily complicated. In a special limiting case Helmholtz's theory becomes considerably simplified, and its equations in this case become the same as those of Maxwell's theory; only one form of the force remains, and this is propagated with the velocity of light. I had to try whether the experiments would not agree with these much simpler assumptions of Maxwell's theory. The attempt was successful. The result of the calculation are given in the paper on 'The Forces of Electric Oscillations, treated according to Maxwell's Theory'."

This paper was published in 1889, one year after the discussion of Hertz's first results, which apparently were not sufficient to conclude which of
the two theoretical descriptions was more adequate. Already discouraged by the complexity of Helmholtz’s approach for a careful account of the experimental data, Hertz tried in this paper to show how the observed singularities in the propagation of the electric force could be described by Maxwell’s theoretical scheme. As Hertz explained [22]: “The results of the experiments on rapid electric oscillations which I have carried out appear to me to confer upon Maxwell’s theory a position of superiority to all others. Nevertheless, I based my first interpretation of these experiments upon the older views, seeking partly to explain phenomena as resulting from cooperation of electrostatic and electromagnetic forces. To Maxwell’s theory in its pure development such a distinction is foreign. Hence I now wish to show that the phenomena can be explained in terms of Maxwell’s theory without introducing this distinction. Should this attempt succeed, it will at the same time any question as to a separate propagation of electrostatic force, which is meaningless in Maxwell’s theory.”

In this famous paper, Hertz wrote Maxwell’s equations in the form in which they are known today (the Hertz-Heaviside form) and also derived the distribution of force lines for the radiating oscillator (Hertz vibrator). In other words, this important contribution to the Faraday-Maxwell field theory consisted in the development of the general source-field relation previously unknown. (Today this method bears Hertz’s name and is based on Fourier analysis of dipole and multi-pole radiation). Using these calculations, Hertz found an explanation (alternative to that based exclusively on Helmholtz’s ideas) of the singularities he had observed in the distribution of radiation in the near field (the apparently instantaneous behaviour of the electrostatic component) In Hertz’s words [23]: “Let us now investigate whether the present [Maxwell’s] theory leads to any explanation of the phenomena... At great distances the phase is smaller by the value $\theta$ than it would have been if the waves had proceeded with constant velocity from the origin; the waves, therefore, behave at great distances as if they had travelled through the first half wavelength with infinite velocity.”

Interestingly, this prediction of Maxwell’s theory concerning the infinite phase-velocity for the near-field zone based on straightforward Fourier analysis (Hertz’s method) appears in the modern literature [24]. Surprisingly, however, there is no interpretation of it in the conventional text-books. Hertz himself paid no attention to this prediction beyond the fact that it gave him a new interpretation of his results, different from that provided by the Helmholtzian approach. It is possible that Hertz did not realise (or had no time to realise) all the conceptual implications of the new prediction, which has no clear meaning in the framework of the Faraday-Maxwell contact-field doctrine. The prediction would imply the existence of a small but macroscopic region where the notion of Faraday locality becomes invalid. On the other hand, it also shows some possible ‘fuzziness’ in the relationship between static and dynamic limits in Maxwell’s theory, as already mentioned earlier. Bearing that ambiguity in mind, it would not be surprising if Maxwell’s theoretical predictions for static and quasistatic phenomena were found to be similar to the older IAAAD views. It is obvious that phenomena in the near field zone (less than half wavelength) should be regarded as quasi-static in the Hertzian analysis and therefore, implicit time dependent that is not Maxwellian but Helmholtzian feature for longitudinal components. This reasoning should have cast doubt on Hertz’s explanation of the experimental results as not being completely in the spirit of the Faraday-Maxwell’s conceptual foundations. It is surprising, then, that almost no-one seemed to have been worried by the presence of non-Faraday’s elements in Maxwell’s approach. By the same token, it is no less surprising that Hertz’s explanation was so unconditionally accepted by Maxwell’s followers. But although Hertz was satisfied that his calculations had accounted for the majority of the observed phenomena, he stressed that he had not succeeded in removing all the difficulties from his experimental verification of Maxwell’s theory. He confessed that [25]: “…I hoped to be able to devise some way of making observations on waves in free air, that is to say, in such a manner that any disturbances which might be observed could in no wise be referred to any action-at-a-distance. This last hope was frustrated by the feebleness of the effects produced under the circumstances.”

Although Hertz’s satisfaction with Maxwell’s theory was understandable, there were still insufficient arguments for making a truly decisive choice, bearing in mind that the Helmholtzian approach remained in qualitative agreement with the observed singularities. However, no further experiments or calculations for testing the quantitative predictions of Helmholtz’s theory have been attempted. In our retrospective view of these stated reservations of Hertz, as well as of so many other thinkers of the time, the unconditional acceptance with which Hertz’s experiments and their interpretations were received might seem somewhat unjustified. Hertz himself did not expect such support and attributed it to the heavy philosophical burden of the old and unresolved dilemma of the choice between the IAAAD and contact-action doctrines.

Another detail which may also have contributed to the unconditional approval of Hertz’s results among the scientific community was, as already stated, the rather misleading title of his paper on the finite propagation of electromagnetic interactions, possibly due to unawareness of Helmholtz’s classification of electrostatic and electromagnetic forces. However, in
contrast to this general enthusiasm for Hertz’s results, Helmholtz’s supporters adopted rather lukewarm attitude, although there was also some strong opposition from such as P. Duhem, an eminent French mathematician, physicist and philosopher of science at the beginning of the twentieth century. He was one of a small group of scientists who refused to accept Hertz’s experiments as conclusive. Moreover, Duhem was the first to raise doubts about the whole concept of ‘crucial’ experiments [26]. A good mathematician and outspoken critic of the inconsistencies in Maxwell’s theory, he became one of the principle advocates of Helmholtz’s approach [27]: "...Physicists are caught in this dilemma: Abandon the traditional theory of electric and magnetic distribution (electro- and magnetostatics), or else give up the electromagnetic theory of light. Can they not adopt a third solution? Can they not imagine a doctrine in which there would be a logical reconciliation of the old electrostatics, of the old magnetism, and of the new doctrine that electric actions are propagated in dielectrics? This doctrine exists; it is one of the finest achievements of Helmholtz; the natural prolongation of the doctrines of Poisson, Ampere, Weber and Neumann, it logically leads from the principles laid down at the beginning of the nineteenth century to the most fascinating consequences of Maxwell’s theories, from the laws of Coulomb to the electromagnetic theory of light; without losing any of the recent conquests of electrical science, it re-establishes the continuity of tradition.”

However, it appears that this call of Duhem’s for a ‘third solution’ fell mainly on deaf ears. So, whilst appreciating the difficulties of Hertz’s pioneering investigations, and taking into consideration his struggle through the uncertainties and controversies of the electrodynamics of his time, the fact remains that his final opting for Maxwell’s theory was not based on strict scientific logic. What needs to be done, therefore, is clearly to identify the criteria for acceptance and apply these to the existing alternative theories. The detailed examination of how and why a certain theory is confirmed or refuted by experimental tests is, of course, a matter of the methodology and philosophy of science. Hertz was apparently not fully aware of the need to test his choice from this systematic, methodological and philosophical standpoint.

As did the majority of his contemporaries, Hertz intuitively applied a criterion of empirical verification, in the hypothetico-deductive manner prevailing in 19-th century science. This method consisted of creating hypotheses in the form of postulates and then making deductions from these which could be either confirmed or rejected by experiment. However, from the modern methodological standpoint, as it is now well recognised, empirical verification is a condition that is necessary but not sufficient for establishing the truth of a theory. The fact is that an empirical observation may ‘verify’ any number of different, yet equally valid theories sharing that same prediction. The only reliable way, therefore, of deciding between theories is not to ‘verify’ any one of them in particular, which might just as well verify any number of them but, if possible, to eliminate all but one of the contenders. The only way of doing this is, of course, by the method of refutation. This method may be carried out logically, by pure ratiocination (as by pointing out some logical or mathematical contradiction) or by demonstrating that empirical predictions made by some logically consistent theory are false. However, even if a theory is logically sound, if it makes no falsifiable predictions, it cannot be refuted. Such a theory, according to Popper, cannot qualify as a scientific theory. This criterion, introduced by Popper in the late 1920s, thus provided a reliable criterion for separating genuine scientific theories from pseudo-scientific or ‘metaphysical’ theories by which, Popper meant theories which make no predictions that can, even in principle, be empirically falsified (it appears to be the case of some of the modern quantum field theories such as quantum gravity, string theory etc.).

It may be of interest, then, to consider how this more modern criterion of falsifiability might have been applied to Hertz’s decision, had it been available at the time. Qualitatively, as we have seen, both Maxwell’s and Helmholtz’s theories fit equally well all the observations made by Hertz, namely:

1. material changes in the surrounding space;
2. finite propagation of transverse components with the speed of light;
3. empirical observation of energy between interacting bodies etc.

The present-day scientific method suggests that the next step is to explore the difference between the experimental predictions of the two theories, beyond those that are already known, and to separate-out the different, non-compatible but empirically verifiable predictions that is, to determine which of them, if any, are sufficient to explain all the known facts, as distinct from those that are merely necessary. Thus, in the case of alternative electrodynamics theories, the core of the new ‘crucial’ experiment should be to test, experimentally, statements specifically describing the character of the longitudinal electric components which distinguish the Helmholtzian from the Maxwellian theory. With regard to the latter, no decisive, unambiguous information of the kind necessary to refute Helmholtz’s theory has yet been found in Hertz’s experimental papers. As a result, by this same reasoning Helmholtz’s theory could not be conclusively ruled out, since it remained perfectly falsifiable and did not contradict any known observational fact.
4. Arguments in favour of alternative Helmholtz-type theory of electromagnetism

As already mentioned, the criterion of falsifiability requires any truly scientific alternative theory to be logically self-consistent. Logical inconsistencies lead to bogus predictions. In this way it is interesting to remind that several aspects of standard conventional electrodynamics are found to be unsatisfactory, despite all the advances claimed by relativity and quantum mechanics. Conventional electrodynamics is thus still not free from untreatable inconsistencies, as in its implications regarding self-interaction, infinite contribution of self-energy, the concept of electromagnetic mass, indefiniteness in the flux of electromagnetic energy, etc. These internal difficulties explain why, from the beginning to the middle of the 20-th century, there were unceasing efforts to modify either Maxwell’s equations or the underlying conceptual premises of electromagnetism. The present status of classical electrodynamics can be expressed by words of R. Feynman [28]: "...this tremendous edifice [classical electrodynamics], which is such a beautiful success in explaining so many phenomena, ultimately falls on its face. When you follow any of our physics too far, you find that it always gets into some kind of trouble. ...the failure of the classical electromagnetic theory. ...Classical mechanics is a mathematically consistent theory; it just doesn’t agree with experience. It is interesting though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself. There are difficulties associated with the ideas of Maxwell’s theory which are not solved by and not directly associated with quantum mechanics...”.

One of the latest systematic accounts of these difficulties has been made recently in [29-31]. In particular, the old problem of Maxwell’s electrodynamics, concerning the uncertain relationship between static and dynamic limits, has been brought into greater relief. Pure mathematical analysis [29] has shown that the conventional theory does not ensure a continuous transition between static and dynamic limits, which is, surely, in itself, strange to contemplate in the context of the Faraday-Maxwell continuous-field concept. Interestingly, it has also been found that if the condition of continuous transition between static and dynamic limits is imposed explicitly in mathematical terms, then conventional boundary condition should be replaced by generalised boundary conditions. As a result, the structure of general solutions to Maxwell’s equations have to be modified to include non-local longitudinal components. It is interesting to note that the possible difficulty with conventional boundary conditions in the classical field theory was realised by A. Einstein himself a few month before his death in 1955. In the last edition of The Meaning of Relativity he added the following [32]: "...A field theory is not yet completely determined by the system of field equations: Should one postulate boundary conditions?: Without such a postulate, the theory is much too vague. In my opinion the answer to the question is that postulation of boundary condition is indispensable.”

Thus, a different choice of a boundary condition can result in a different approach (Maxwellian or Helmholtzian) in the framework of the same system of Maxwell’s equations (a limit case of Helmholtz’s equations). It, therefore, results in a different time dependence of forces: explicit time dependence for longitudinal and transverse components in Maxwell’s theory and explicit/implicit time dependence for transverse/longitudinal components in Helmholtz’s theory. As far as it is reflected in the history of electrodynamics literature, Hertz did not attribute any importance to the choice of boundary conditions and believed that the same differential equations should have only one basic interpretational background.

In respect to the longitudinal components of electromagnetic field, it is noteworthy that they turn out to be in agreement with the requirements of relativistic invariance in the neo-Helmholtzian approach [29-31]. Moreover, they behave in the same manner as it is accepted by conventional relativistic theory for fields and potentials of one uniformly moving charge under Lorentz relativistic transformations. With regard to the latter, it is well-known that field lines of one uniformly moving charge are radial, i.e. exhibit IAAAD features. However, in the traditional interpretation, this radial character of field-lines is considered fictitious (as a class of legitimate singularities), whereas in the Helmholtz-type approach developed in [29-31] it is a perfectly realistic representation. Thus, contrary to what could have been expected from the point of view of the conventional electrodynamics, instantaneous longitudinal fields can behave in accordance with required Lorentz relativistic symmetry.

In the approach reviewed below, longitudinal components do not take part in a local energy transfer avoiding possible conflict with the special relativity theory. A local energy transfer is an exclusive prerogative of transversal components associated with radiation. In fact, it is well-known that Einstein’s theory does not limit phase velocities, if there is no local energy transfer. Thus, a purely mathematical analysis of a self-consistency of electromagnetic theory with required relativistic properties leads automatically to the alternative approach very similar by spirit to the Helmholtzian doctrine based on a superposition of local transversal and instantaneous longitudinal forces.

In summarising this discussion it is interesting to note another possible attractiveness of Helmholtz’s
Within Helmholtz’s foundations it is even possible to formulate a classical model of the atom contrary to a formulation of non-radiation condition in [29]. Components are responsible for the appearance of electromagnetic radiation potential theory with bipartite type of interactions. This fundamental conflict is such that, for some people, the only way of resolving it seems to be to combine or superimpose these incompatible requirements for locality and non-locality in some purely expedient and compromising way. In view of these reasoning it also can be suggested that any physical theory of interaction might not be pure conventional field theory (in the sense of local field) but be complemented by non-local longitudinal components (in the sense of non-local potential field).

In modern physics, as it is well-known, longitudinal components are responsible for the appearance of infinities that in special cases and with enormous difficulties can be eliminated by mathematical means, leaving nevertheless some kind of conceptual dissatisfaction. In some cases such as quantum gravity the infinities resist to be removed indicating that there is no unified conceptual approach to the infinities problems and it appears to have more mathematical convenience than physical justification. In Helmholtz’s and neo-Helmholtz’s theory [29–31] with the clear distinction between transverse and longitudinal component, the latter one does not contribute infinite amount of self-energy as it does not in every classical (newtonian) non-local potential theory with bipartite type of interactions. This property of longitudinal components allowed a formulation of non-radiation condition in [29] which states that “...a limited class of motion exists when accelerated charged particles do not produce electromagnetic radiation.” It means that within Helmholtz’s foundations it is even possible to formulate a classical model of the atom contrary to Maxwell’s electrodynamics where all such attempts failed. All these developments will be reviewed in this work.

Thus, as the relevant historical literature shows, Helmholtz’s foundations promise an altogether more logical solution to fundamental problems of modern physics. This may suggest a way of reconciling classical electrodynamics and quantum mechanics in a less ad hoc and altogether more rational way than has, up till now, seemed obligatory. Recent experimental confirmations of the violation of Bell’s inequalities in quantum mechanical measurements and entanglement in quantum optics shed some light on a possible alternative foundations of classical electrodynamics incorporating IAAAD longitudinal components in the same framework of Maxwell’s equations.

5. An example of ambiguity of Faraday-Maxwell’s concept of local (contact) field in the conventional electromagnetic theory

At the beginning of this Section we show one of the confusions of classical electrodynamics in describing electromagnetic field of an accelerated charge. The attractiveness of this example consists in the way it lightens the main conventional theory difficulties and the way it leads to the Helmholtzian-type foundations of classical electrodynamics. Let us consider a charge e moving in a laboratory reference system with constant acceleration \(a\) along the positive direction of the \(X\)-axis. An electric field created by an arbitrarily moving charge is given by the following expression obtained directly from Lienard-Wiechert potentials [33]:

\[
\mathbf{E}(\mathbf{R}, t) = e \left( \frac{\mathbf{R} - \mathbf{R}_v}{c} \right)^3 \left( 1 - \frac{v^2}{c^2} \right) + e \left( \frac{\mathbf{R}_v}{c} \right) \mathbf{v} \left( \frac{\mathbf{R} - \mathbf{R}_v}{c} \right)^3. \tag{1}
\]

All values in the right-hand are taken in the moment of time \(t_0 = t - \tau\), where \(\tau\) is the retarded time. Since all vectors are collinear, the second term in (1) is zero. In the conventional theory, the Poynting vector represents electromagnetic field energy flow per unit area per unit time across a given surface,

\[
\mathbf{S} = \frac{c}{4\pi}[\mathbf{E}, \mathbf{H}], \quad \mathbf{P} = \frac{1}{c^2}\mathbf{S}, \tag{2}
\]

where \(\mathbf{S}\) is the Poynting vector, \(\mathbf{P}\) is momentum density, and \(\mathbf{E}\) and \(\mathbf{B}\) are electric and magnetic field strengths, respectively. Analysing (2), one can easily note that \(\mathbf{S}\) and \(\mathbf{P}\) (and, therefore, all electromagnetic energy flow) are exactly zero (\(\mathbf{S} = 0\)) along the \(X\)-axis. On the other hand, from the energy conservation law,

\[
W = \frac{E^2 + B^2}{8\pi}, \quad \frac{\partial W}{\partial t} = -\nabla \mathbf{S}, \tag{3}
\]

we conclude that \(W\) and \(\partial W/\partial t\) should differ from zero everywhere along \(X\)-axis because there is a linear relationship between \(W\) and \(E^2\) changing in time along \(X\)-axis. An ambiguity takes place if, for instance, the charge is moving in some arbitrary way along the \(X\)-axis. As a result the energy density \(W\) should also alter as a function of changing electric field \(E\). Then the question logically arises: what is the mechanism...
that changes electric and magnetic field components at some fixed distance from the charge on X-axis if there is apparently no electromagnetic field energy transfer in that direction ($S = 0$)? This ambiguity is due to the fact that in the conventional theory based on the concept of local (contact) field, which energy has to be stored locally in space, any change of field components is indistinguishable without field energy flux. This is obviously violated in the above-mentioned example that brings into question an assumed sufficiency of transverse solutions alone to describe all properties of electromagnetic field. At least, the resolution of this problem of longitudinal components in the conventional theory? In the following we will try to show that the problem of longitudinal components is underestimated in classical electrodynamics (perhaps by historical reasons). There should be a change of attitude towards its status. Mathematical and physical reasons in favour of Helmholtz-type foundations will be given to show a paramount importance of longitudinal components to build up a self-consistent classical electrodynamics and its possible reconciliation with quantum mechanics.

6. Mathematical foundations of Helmholtz-type electrodynamics

Let us recall that a complete set of Maxwell’s equations in vacuum is

\[ \nabla \times \mathbf{E} = 4\pi \rho, \]
\[ \nabla \times \mathbf{B} = 0, \]
\[ \nabla \cdot \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \]
\[ \nabla \cdot \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}. \]

If this system of equations is really complete and boundary conditions are adequate, it should describe all electromagnetic phenomena without exceptions and ambiguities. It is often convenient to introduce potentials, satisfying the Lorentz condition

\[ (\nabla, \mathbf{A}) + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0. \]

As a result, the set of coupled first-order partial differential equations (6)–(9) can be reduced to the equivalent pair of uncoupled inhomogeneous D’Alembert’s equations:

\[ \Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho(r, t), \]
\[ \Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}(r, t). \]

Differential equations have, generally speaking, an infinite number of possible solutions. An uniquely determined solution is selected by laying down sufficient additional conditions. Different forms of additional conditions are possible for the second order partial differential equations: initial value and boundary-value conditions. A general solution of the...
D’Alembert equation is considered as an explicit time-dependent function of the type \( g(R, t) \). Let us discuss a very subtle point related to the use and interpretation of implicit and explicit time dependencies in the conventional electrodynamics. We think that as far as this problem is not cleared up, the classical theory will remain beset of ambiguities. Helmholtz-type approach reviewed below makes that distinction very clear.

Special relativity well established that in the stationary approximation (charge moving with a constant velocity) all fields components are implicit time-dependent functions of the type \( f(R(t)) \). Field lines remain radial in all inertial frames of reference and, hence, depend on the instant position of the charge. As a consequence, time \( t \) is not an independent variable any more in this case and enters as a parameter through space position of the charge \( R(t) \). Hence, the use of partial time derivatives \( \partial / \partial t \), \( \partial^2 / \partial t^2 \) etc. (according to their formal mathematical definition) is inadequate if a function has not two or more independent variables. Nevertheless, in basic texts on classical electromagnetic theory partial time derivatives are indiscriminately applied even for implicit time dependent functions in the proper sense of total time derivatives. (Some clear examples will be done in the next Section discussing the use of continuity equation).

Looking back at D’Alembert’s equations (11),(12), space variable \( R \) should be fixed under the action of partial time derivative \( \partial^2 / \partial t^2 \). Fixing \( R(t) \), means that there is no change with time \( t \) playing the role of a parameter. Thus, partial time derivatives vanish from D’Alembert equation in the case of uniformly moving charge. Poisson’s equation for four-vector \( (\phi, A) \) with implicit time dependence appears to be appropriate one. We especially made a detailed analysis because of a confusion in conventional texts on classical electromagnetism about explicit use of Poisson’s equations for uniformly moving charge (but as we have seen, they do it tacitly). It is commonly thought that only D’Alembert’s equation (i.e. that only D’Alembert’s operator \( \Delta - \partial^2 / \partial t^2 \)) is relativistically invariant under Lorentz’s transformations. As we will discuss later in the Section VII in connection with gauge invariance and summarising it up in the Section VIII, Poisson’s equation in four-vector representation \( (\phi, A) \) (as well as Poisson’s differential operator \( \Delta \)) can also be considered relativistically invariant when applied to implicit time-dependent potentials, reproducing all results of special relativity for inertial frames of reference. Poisson’s differential operator \( \Delta \) is not covariant but invariant under Lorentz’s transformations. Time variable is not any more independent in this case and cannot be used for covariant representation of D’Alembert’s differential operator. It is endorsed by the well-known fact that covariance is not necessary, it is only sufficient for relativistic invariance.

Thus, we can conclude that D’Alembert equations have general solutions in form of explicit time-dependent functions whereas Poisson’s equations have only implicit time dependent solutions. The following question becomes obvious: how any transition from D’Alembert and Poisson’s equations is described in the conventional formalism? As a matter of fact, this question has not even been asked because Poisson’s equation has not been recognised as covering implicit time-dependent phenomena (it was applied exclusively in electro- and magnetostatics with no time dependence at all). This question, unexplored by the conventional approach, contains a very serious difficulty.

As we shall demonstrate below, a continuous transition between solutions of D’Alembert’s and Poisson’s equations, respectively, is not mathematically ensured in classical electromagnetism. Based on the premises of a continuous nature of electromagnetic phenomena, one can assume that any general implicit time solution of Poisson’s equation should be continuously transformed into explicit time solutions of D’Alembert’s equations (and vice versa). This requirement can also be formulated as a mathematical condition on the continuity of general solutions of Maxwell’s equations at every moment of time. By force of the uniqueness theorem for the second order partial differential equations, only one solution exists satisfying given initial and boundary conditions. Consequently, the continuous transition from solutions of D’Alembert’s equation into solutions of Poisson’s equation (and vice versa) should be ensured by the continuous transition between respective initial and boundary conditions. This is the point where the conventional approach fails again. Only implicit time-dependent function \( f(R(t)) \) can be unique solution of Poisson’s equations and boundary conditions for external problem are to be formulated in the infinity. On the other hand, the solution of D’Alembert’s equation is an explicit time-dependent function \( g(R(t), t) \) since only it fits requirements of Faraday–Maxwell’s electrodynamics as a physically sound solution for the notion of local (contact) field. The boundary conditions in this case are given in a finite region. It makes no sense to establish them at the infinity if it cannot be reached by any perturbations with finite speed velocity. As far as one deals with large external region, effects of boundaries are still insignificant over a small interval of time, and, therefore, it is convenient to consider the limiting problem with initial conditions for an infinite region (initial Cauchy’s problem). This is how in mathematical physics areas of infinite dimensions are introduced into consideration.

Let us look carefully at the standard formulation of respective boundary-value problems in a region extending to infinity. There are three external boundary-value problems for Poisson’s equation.
They are known as the Dirichlet problem, Neumann problem and their combination. The mathematical formulation, for instance, for Dirichlet’s boundary conditions requires to find a function \( u(r) \) satisfying [35]:

(i) Laplace’s equation \( \Delta u = 0 \) everywhere outside the given system of charges (currents).

(ii) Solution \( u(r) \) is continuous everywhere in the given region and takes the given value \( G \) on the internal surface \( S: u|_S = G \).

(iii) Solution \( u(r) \) converges uniformly to 0 at infinity: \( u(r) \to 0 \) as \( |r| \to \infty \).

The final condition (iii) is essential for a unique solution! In the case of D’Alembert’s equation the standard mathematical formulation is different. Obviously, we are interested only in the problem for an infinite region (initial Cauchy’s problem). So it is required to find the function \( u(r(t), t) \) satisfying [35]:

(j) homogeneous D’Alembert’s equation everywhere outside the given system of charges (currents) for every moment of time \( t \geq 0 \)

(jj) initial conditions in all infinite regions as follows:
\[
\begin{align*}
  u(r, t) \big|_{t=0} & = G_1(r); \quad u_t(r, t) \big|_{t=0} = G_2(r).
\end{align*}
\]

The condition (iii) about the uniform convergence at infinity is not mentioned. Recall here that Cauchy’s problem is considered when one of the boundaries is insignificant over all time of a process. In conventional electrodynamics it means that any perturbation with finite spread velocity will never reach the limits of the region under consideration during the time of observation. From the conventional point of view, condition (iii) formally included into Cauchy’s problem can never affect the solution and, hence, might not be taken into account seriously for selecting of adequate solutions. In fact in the context of local field, the inclusion of the condition (iii) becomes meaningless since only explicit time-dependent solutions (retarded waves with finite spread velocity) are allowed by conventional electrodynamics to solutions of D’Alembert’s equation.

On the other hand, we underline here that the absence of the condition (iii) for every moment of time in the standard mathematical formulation of Cauchy’s initial problem does not ensure the continuous transition into external boundary-value problem for Poisson’s equation and, as a result, mutual continuity between the corresponding solutions cannot be expected by force of the uniqueness theorem. This unambiguous mathematical fact should be considered as one of the most warning signals of possible flaws in the mathematical formalism of contemporary Maxwell’s electrodynamics. The only way that seems to be obligatory to satisfy the property of continuity of electromagnetic field (in other words, to keep the continuity in transition between solutions of D’Alembert and Poisson’s equations), is the inclusion of the condition (iii) for every moment of time in the standard mathematical formulation of Cauchy’s initial problem. It obviously ensures the continuous transition into external boundary-value problem for Poisson’s equation (and vice versa) and implies a structure of a general solution as a superposition of separate non-reducible to each other functions of the type
\[
f(R(r)) + g(r, t).
\]  

When applied to potentials, this statement takes a form:
\[
\begin{align*}
  \varphi(r, t) &= \varphi_0(R(t)) + \varphi^*(r, t), \\
  A(r, t) &= A_0(R(t)) + A^*(r, t),
\end{align*}
\]

where for one charge system \( R(t) = r - r_q(t) \); \( r \) is a fixed distance from the point of observation to the origin of the reference system and \( r_q(t) \) is the position of the charge at the instant \( t \).

The presence of the condition (iii) in the formulation of Cauchy’s problem turns out to be meaningful for any moment of time, and the corresponding boundary conditions keep continuity in respect of mutual transformation. That makes the condition (iii) irremovable from the formulation of initial Cauchy’s problem resulting in fundamental (irremovable) nature of implicit time-dependent (or longitudinal) components \( \phi_0 \) \( A_0 \) responsible for the interparticle interaction. Potentials with explicit time-dependence \( \phi^* \) and \( A^* \) vanish in the steady-state case, leaving only explicit time-dependent functions \( \phi_0 \) \( A_0 \) in the total potential (left-hand side of (14) and (15)). Now, contrary to the conventional approach, it is clear how the total solution \( \phi \) (or \( A \)) in left-hand side of (14),(15) with explicit time dependence undergoes transformations into solution with implicit time dependence (and vice versa).

Faraday-Maxwell’s approach does not allow to take into account the first term in right-hand side of (14),(15) as full-value part of any general solution. Turning to the above-mentioned ambiguity at the beginning of the previous section, we see now that the novel solution in form of (14),(15) can describe the change of electric field component along the X-axis at any distance and at any time. It casts doubts on the general belief that Lienard- Wiechert potentials (as only explicit time-dependent solutions of D’Alembert’s equations for Cauchy’s problem) should be considered as unique general solutions to Maxwell’s equations regardless the context of boundary conditions. In connection to the latter it is worth reminding Einstein’s words (see the Section III) that field theory is not only determined by the system of field equations but also by postulation of boundary conditions. In fact, Lienard and Wiechert formulated the initial Cauchy problem.
for electromagnetic components several years before the appearance of Einstein's principle of relativity. Thus, a priori imposed boundary conditions were not assumed to have adequate relativistic properties. This is another open question in the conventional approach whether relativistic requirements should be reflected in the mathematical formulation of the initial boundary problem. In this respect, we only stress that additional condition (iii) is such an invariant because it is irremovable and unchangeable in every frame of reference.

Let us consider again a pair of uncoupled inhomogeneous D’Alembert’s equations (11), (12) with initial conditions (j), (jj) and (iii). For some purposes, it is convenient to decompose (11), (12) into two pairs of second order differential equations for each component of general solution (14), (15):

\[ \Delta \varphi_0 = -4 \pi \rho(r, t), \]
\[ \Delta A_0 = -\frac{4\pi}{c} j(r, t), \]

and

\[ \Delta \varphi^* = -\frac{1}{c^2} \frac{\partial^2 \varphi^*}{\partial t^2} = 0, \]
\[ \Delta A^* = -\frac{1}{c^2} \frac{\partial^2 A^*}{\partial t^2} = 0, \]

with initial and boundary conditions given, for instance, in the case of electric potential. The equation (16), apart from (iii), is supplemented by

\[ \varphi_0(r)|_{t=0} = G \]

whereas (18) has to be added with

\[ \varphi^*(r, t)|_{t=0} = G_1 - \varphi_0(r)|_{t=0}, \]
\[ \varphi^*_t(r, t)|_{t=0} = G_2 - \frac{d}{dt}\varphi_0(r)|_{t=0}. \]

In the theory of differential equations any complete solution of (11), (12) consists of a general solution of homogeneous D’Alembert’s equation plus some particular solution of the inhomogeneous one. Thus, we can assume that the same procedure can be applied to its equivalent formulation in form (16)–(19). On one hand, a complete solution should be formed by two independent general solutions satisfying homogeneous Poisson’s and homogeneous wave equations, respectively, and, on the other hand, it has to include one particular solution (as a linear combination of non-reducible components (14), (15), satisfying inhomogeneous D’Alembert’s equations (11), (12). Relationship between both components (longitudinal and transverse) of electromagnetic field is governed by (21) and (22) and is contained in the particular solution of inhomogeneous D’Alembert’s equations. A more comprehensive study of the matter will be done elsewhere.

Thus, the initial set of Maxwell’s equations has been decomposed into two pairs of equations with independent general solutions for each pair that are coupled only through the partial solution of the whole set of equations (16)–(19) or (11), (12). The first pair (16), (17) manifests the instantaneous and longitudinal aspect of electromagnetic interactions (action-at-a-distance) while the second one (18), (19) characterises explicit time-dependent phenomena related to the propagation of transverse waves (light, radiation etc.). It is obvious thus that Helmholtz’s basic ideas are fundamentally compatible with Maxwell’s equations.

The potential separation (14), (15) implies the same procedure with respect to the field strengths,

\[ E(r, t) = E_0(R(t)) + E^*(r, t), \]
\[ B(r, t) = B_0(R(t)) + B^*(r, t), \]

where \( E_0 \) and \( B_0 \) are instantaneous longitudinal fields.

To finish this Section we would like to mention that Villecco’s independent analysis endorsed our claims on discontinuity problem in the classical electromagnetic theory. He found that [36]: "...the transition between two different states of uniform velocity via an intermediate state of acceleration results in a type of discontinuity in functional form: Though no known law is violated in this processes, there is a sense of intrinsic continuity which is nevertheless violated...".

7. Mathematical inconsistencies in the formulation of Maxwell-Lorentz equations for one charge system

Let us come back again to the original set of Maxwell’s equations (6)–(9) for the reference system at rest supplemented by the continuity equation

\[ \frac{\partial \rho}{\partial t} + (\nabla, j) = 0. \]

In the phenomenological theory of electromagnetism the hypothesis about the continuous nature of the medium was one of the foundations of Maxwell’s theoretical scheme. This point of view succeeded in uniting so many electromagnetic phenomena without the necessity to consider a specific structure of matter. Nevertheless, a macroscopic character of the charge conception defines all well-known limitations on Maxwell’s theory. For instance, the system of equations (6)–(9) in a steady state approximation corresponds to a quite particular case of continuous and closed conduction currents (motionless as a whole).

In 1895, the theory was extended by Lorentz for a system of charged particles moving in vacuum. Since
then it has been widely assumed that the same basic laws are valid microscopically as it is macroscopically in the case of original Maxwell’s equations. This means that in Lorentz form all macroscopic values of charge and current densities have to be substituted by their microscopic values. Let us write explicitly the Lorentz field equations for one point-charged particle moving in vacuum [33]:

\[
(\nabla, \mathbf{E}) = 4\pi q\delta(\mathbf{r} - \mathbf{r}_q(t)),
\]

\[
(\nabla, \mathbf{B}) = 0,
\]

\[
[\nabla, \mathbf{H}] = \frac{4\pi}{c} q\mathbf{v}\delta(\mathbf{r} - \mathbf{r}_q(t)) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},
\]

\[
[\nabla, \mathbf{E}] = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},
\]

where \(\mathbf{r}_q(t)\) is the coordinate of the charge at the moment of time \(t\).

In order to achieve a complete description of a system consisting of fields and charges in the framework of electromagnetic theory, Lorentz supplemented (26)–(29) by the equation of motion:

\[
\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{v}, \mathbf{B}],
\]

where \(\mathbf{p}\) is the momentum of the particle.

The equation of motion (30) introduces an expression for the mechanical force known as Lorentz force which in the electron theory formulated by Lorentz has a clear axiomatic and empirical status. Later on we shall discuss some disadvantages related with the adopted status of the Lorentz force conception.

Macroscopic Maxwell's equations (6)–(9) may be obtained now from Lorentz’s equations (26)–(29) by some statistical averaging process, using the structure of material media. The mathematical language for equations (26)–(29) is nowadays widely accepted in the conventional classical electrodynamics. However, there is an ambiguity in the application of these equations to the case of one uniformly moving charge. A simple charge translation in space produces alterations of field components. Nevertheless, they cannot be treated in terms of Maxwell’s displacement currents. Strictly speaking, in this case all Maxwell’s displacement currents proportional to \(\partial \mathbf{E}/\partial t\) and \(\partial \mathbf{B}/\partial t\) vanish from (28)–(29). This statement can be reasoned in two different ways:

1. \(\partial \mathbf{E}/\partial t = 0\) and \(\partial \mathbf{B}/\partial t = 0\), since all field components of one uniformly moving charge are implicit time-dependent functions (time does not enter as an independent parameter but only through space variable) so that from the mathematical standpoint only total time derivative makes sense in this case whereas partial time derivative turns out to be not adequate (time and distance are not independent variables);

2. a non-zero value of \(\partial \mathbf{E}/\partial t\) and \(\partial \mathbf{B}/\partial t\) would imply a local variation of fields in time regardless any change in the position of the charge (space co-ordinate is fixed when partial time derivative is taken) and, hence, would imply the propagation of those local variations in form of transverse electromagnetic waves.

This would strongly contradict the well-established in special relativity fact that one uniformly moving charge does not produce any electromagnetic radiation at all.

Thus, a mathematically rigorous interpretation of (28),(29) in the case of a charge moving with a constant velocity leads to the following conclusion: in a charge-free space the value of \(\partial \mathbf{E}/\partial t = 0\) and, therefore, the value of \(\text{rot} \mathbf{H}\) is also equal to zero in free space

\[
[\nabla, \mathbf{H}] = \frac{4\pi}{c} q\mathbf{v}\delta(\mathbf{r} - \mathbf{r}_q(t)).
\]

On the other hand, field components of one uniformly moving charge can be treated exactly in the framework of Lorentz’s transformations. Therefore, for any purpose exact relativistic expressions for electric and magnetic fields and potentials should be applied [33]

\[
\mathbf{E} = q \frac{(1 - \beta^2)(\mathbf{R} - R\tilde{\beta})}{(\mathbf{R} - R\tilde{\beta})^3},
\]

\[
\mathbf{H} = \frac{1}{c}[\mathbf{v}, \mathbf{E}],
\]

where \(\tilde{\beta} = \mathbf{v}/c\).

Thus, we arrive here at the important conclusion: generally speaking, according to special relativity theory the value of \(\text{rot} \mathbf{H}\) is not equal to zero in any point out of moving charge and takes a well-defined value

\[
[\nabla, \mathbf{H}] = \frac{1}{c}[\mathbf{v}, \mathbf{E}].
\]

For instance, this gives immediately a non-zero value of \(\text{rot} \mathbf{H}\) along the direction of motion (X-axis):

\[
[\nabla, \mathbf{H}](x, x > x_0) = q \frac{2\beta(1 - \beta^2)}{(1 - \beta)^3(x - x_0)^3}.
\]

The conflict with the previous statement of the equation (31) is inevitable. In order to obtain adequacy between the set of field equations (26)–(29) and their relativistic solutions in the case of uniformly moving charge, it is necessary to consider an additional term like that considered in (34). As will be shown in continuation, this assumption for static and quasi-static fields is a supplement of Maxwell’s displacement currents introduced for explicitly time varying fields (explanation of the light as the propagation of transverse electromagnetic waves).
As it is well-known, the necessity of Maxwell’s displacement current was realised on the basis of the following formal reasoning. In order to make equation (8) consistent with the electric charge conservation law in form of continuity equation (25), Maxwell supplemented (8) with an additional term. However, for stationary processes, as we already have seen, this term disappears and equation (8) becomes consistent only with closed (or continuous going off to infinity) currents

\[ (\nabla, j) = 0. \]  

(36)

It is also a direct consequence of continuity equation (25) in any stationary state when all magnitudes have to be treated as implicit time-dependent functions. Thereby, we meet here another difficulty of Lorentz’s equations: uniform movement of a single charged particle (as an example of open steady current), generally speaking, does not satisfy the limitations imposed by (36). It implies some additional term to be taken into account in (36) to fulfill Maxwell’s hypothesis on the circuital character of total currents (conduction plus displacement currents).

Let us have a close look on the continuity equation and its conventional interpretation. In developing the mathematical formalism of his theory Maxwell adopted Faraday’s idea of field tubes for electric and magnetic fields as well as for electric charge flow (conduction currents). As a consequence, in accordance with hydrodynamics language, the continuity equation was accepted as valid to express the hypothesis that a net sum of electric charges could not be annihilated. In this case, the continuity equation reproduces the charge conservation law in the given fixed volume

\[ \frac{dQ}{dt} = \int \int \left\{ \frac{\partial \rho}{\partial t} + (\nabla, j) \right\} dV = 0 \]  

(37)

or in the form of a differential equation

\[ \frac{\partial \rho}{\partial t} + (\nabla, j) = 0, \quad \left( \frac{dQ}{dt} = 0 \right). \]  

(38)

It should be remarked that equation (38) describes exclusively the conservation but not the change of the amount of charge (or matter) in the given volume \( V \). In many scientific writings on electromagnetic theory there is no clear distinction between these two aspects.

If one wants to describe the change of the given volume \( V \), the equation (37) should be replaced by a balance equation (see, for instance, [37])

\[ \frac{dQ}{dt} = \frac{d}{dt} \int \int \rho dV = - \int \int (\nabla, j) dS, \]  

(39)

where \( j \) is a total current of electric charges through a surface \( S \) that bounds the given volume \( V \). In the mathematical language common to all physical theories it means that the rate of increase in the total quantity of electrostatic charge within any fixed volume \( V \) is equal to the excess of the influx over the efflux of current through a closed surface \( S \). On contracting the surface to an infinitesimal sphere around a point one can arrive at the differential equation [37]

\[ \frac{dp}{dt} + (\nabla, j) = 0, \quad \left( \frac{dQ}{dt} \neq 0 \right). \]  

(40)

The balance equation (40) covers the continuity equation (38) as a particular case in which the amount of something (charge or matter) is kept constant in \( V \) during the course of time. Earlier we mentioned that a single charge in motion, generally speaking, could not be treated in terms of the continuity equation (38). When the particle leaves the given volume, it violates locally the charge conservation, invalidating the continuity equation (38). Instead of it the balance equation (40) has to be used. One simple method to prove that is to consider again the example of point-charge moving with a constant velocity. In particular, the charge density is assumed to have implicit time dependence as follows

\[ \rho(r, r_q(t)) = q \delta(r - r_q(t)). \]  

(41)

where \( r \) is a fixed distance from the point of observation to the origin of the reference system at rest; \( r_q(t) \) and \( v_q = dr_q/dt \) are the distance and the velocity of the charge at the instant.

It is easy to show that the total density derivative with respect to time consist of the convection term only, since time enters in equation (41) as a parameter \((\partial \rho/\partial t = 0)\):

\[ \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \left\{ \frac{d}{dt} (r - r_q(t)) \right\} \nabla \rho = - (v_q, \nabla \rho). \]  

(42)

Thus, the balance equation for a single charged particle is fulfilled directly:

\[ -(v_q, \nabla \rho) + \nabla (\rho v_q) = -(v_q, \nabla \rho) + (v_q, \nabla \rho) = 0. \]  

(43)

The next step is to analyse equation (43) in terms of Maxwell’s hypothesis in respect to the circuital character of the total electric current (including displacement current). In other words, the total current of one uniformly moving charge has to be formed by two contributions: the motion of the charge itself (conduction current) and displacement current in outer space:

\[ (\nabla, (j_{\text{cond}} + j_{\text{displ}})) = 0, \]  

(44)

where \( j_{\text{cond}} \) and \( j_{\text{displ}} \) are conduction and displacement currents, respectively.
Thus, we can rewrite (40) in the form of equation (44)

\[
(\nabla, j_{\text{displ}}) = \frac{d}{dt} \left( \frac{1}{4\pi} (\nabla, E) \right) = \left( \nabla, \frac{1}{4\pi} \frac{dE}{dt} \right). \quad (45)
\]

It may be easily verified that two field operations \( \nabla \) and \( d/dt \) are completely interchangeable in (45). Thus, for general motion of the charge when one can disregard its size, Maxwell’s condition on a total current takes the following form (see for the sake of comparison the formula (42)) taking into account the standard expansion of the total time derivative:

\[
(\nabla, j_{\text{displ}}) = \frac{1}{4\pi} \left( \nabla, \left\{ \frac{\partial E}{\partial t} - (v, \nabla_v)E - (a, \nabla_v)E - \ldots \right\} \right), \quad (46)
\]

here \( a \) is acceleration and further terms correspond to derivatives of non-uniform acceleration.

So far we have made use of the formal mathematical approach without any physical interpretation. More specifically, in calculating the full time derivative of \( E \), the convective term (second right-hand term in (46)) should be considered as implicit time-dependent (time variable is fixed when space partial derivative is taken) in agreement with the mathematical definition of partial derivatives. In mathematical language it means that all field alterations produced by a simple charge translation (convective part of the total derivative) take place at the same time in every space point (i.e. instantaneously). This interpretation has no precedents in conventional classical electrodynamics for the case of arbitrary motion whereas for uniformly moving charge this description is the only possible formalism (in special relativity field lines of uniformly moving charge remain radial, i.e. exhibit no retardation in respect to the space position of the charge). Turning back to (46), it is clear that the first right-hand term with partial time derivative describes explicit time-dependent phenomena. Thus, in the same way as it was independently concluded in the Section VI, all field components can be split up into two independent classes with explicit \( E^* \) and implicit \( E_0 \) time dependencies, respectively:

\[
\frac{dE}{dt} = \frac{\partial E^*}{\partial t} - (v, \nabla_v)E_0 - (a, \nabla_v)E_0 - \ldots \quad (47)
\]

A general expression of full displacement current is then taken by the formula:

\[
j_{\text{displ}} = \frac{1}{4\pi} \frac{\partial E^*}{\partial t} - \frac{1}{4\pi} (v, \nabla_v)E_0 - \frac{1}{4\pi} (a, \nabla_v)E_0 - \ldots \quad (48)
\]

Let us stress here one subtle point which will be indispensable in the following discussion of relativistic invariance properties of the present Helmholtz-type approach. The derivation of (47) has considered the partial time derivative to be independent from the space derivative in full agreement with the mathematical formalism of partial derivatives. Thus, the time parameter of implicit time-dependent components (let us call it \( t \)) comes into consideration as an afterthought through the space variable \( R(t) \) and, therefore, can be, in principle, considered as independent from the time variable of explicit time-dependent components (in special relativity this is the so-called proper time \( \tau \)). As we will discuss later, special relativity does not distinguish these two time dependences and tacitly implies \( t = \tau \) that leads to the Lorentz invariance of electromagnetic field components.

In order to come back to the previous discussion of the displacement current concept, let us remind that our initial aim was to find a reasonable form for Maxwell’s circuital condition (44). It would allow to relate field alterations in free space produced by one moving charge with the Maxwell conception of displacement current. From the standpoint of conventional classical electrodynamics, the first term represents the well-known Maxwell displacement current coming up only in non-steady processes whereas the second term can be interpreted only as quasistationary due to its dependence on a charge translation in space (with time as implicit parameter). Further, we will call that term as “convection displacement current”. By the same token, the third right-hand term is due to uniform acceleration and could be called “uniform acceleration displacement current” etc.

The above results motivate an important extension of displacement current concept. First, it postulates the circuital character of the total electric current as it was originally assumed by Maxwell. Second, it permits to fulfill the circuital condition for non-steady as well as for steady processes (static and quasistatic fields), contrary to the conventional approach. Let us give an equivalent mathematical expression of the convection displacement current (in the case of single charged particle):

\[
\frac{1}{c}(v, \nabla)E = \frac{1}{c}v(\nabla, E) - \frac{1}{c}\nabla [v, [v, E]]. \quad (49)
\]

Accordingly, for our purpose we need to remind that in the right-hand side of equation (28) the total current \( (j_{\text{tot}} = j_{\text{cond}} + j_{\text{displ}}) \) must be considered as:

\[
[\nabla, H] = \frac{4\pi}{c}qv\delta(r - r_q(t)) + \frac{1}{c} \frac{\partial E}{\partial t} - \frac{1}{c}v(\nabla, E) + \frac{1}{c}[v, [v, B]] + \ldots \quad (50)
\]
For the sake of simplicity we omit acceleration and other expansion terms in this general formula but they are tacitly implied.

This approach allows the treatment of equation (29) in the same way as (28):

\[
[V, E] = -\frac{\partial B}{\partial t} + \frac{1}{c}(v, \nabla)B = -\frac{\partial B}{\partial t} + \frac{1}{c}v(\nabla, B) - \frac{1}{c}[\nabla, [v, B]] + \ldots
\]

Turning back to the beginning of this Section we note now that for uniform motion \( \text{curl} \, H \) is defined by (50) in every space point out of the charge in the expected way (see (34)). As a final remark, the set of equations (26),(27) and (50),(51) can be regarded as a generalized form of Maxwell-Lorentz system of field equations. In the next section they will be compared with modified Maxwell-Hertz equations extended on one charge system.

8. Reconsidered Maxwell-Hertz theory and relativistically invariant formulation of generalized Maxwell’s equations

Independently of Heaviside, the problem of the modification of Maxwell’s equations for bodies in motion was posed by Hertz in his attempts to build up a comprehensive and consistent electrodynamics [38, 39]. A starting point of that approach was the fundamental character of Faraday’s law of induction represented for the first time by Maxwell in the form of integral equations,

\[
\oint_C \mathbf{H} \, dl = \frac{4\pi}{c} \iint_S j \, dS + \frac{1}{c} \frac{d}{dt} \iint_S \mathbf{E} \, dS, \quad (52)
\]

\[
\oint_C \mathbf{E} \, dl = -\frac{d}{dt} \iint_S \mathbf{B} \, dS \quad (53)
\]

where \( C \) is a contour, \( S \) is a surface bounded by \( C \).

In qualitative physical language Faraday’s law, there must be an electromotive force associated with the traditional non-relativistic treatment of the integral form of Faraday’s law [40]. Namely, if the circuit \( C \) is moving with a velocity \( v \) in some direction, the total time derivative in (52),(53) must take into account this motion (convection derivative) as well as the flux changes with time at a point (partial time derivative) [40],

\[
\oint_C \mathbf{E} \, dl = -\frac{1}{c} \frac{d}{dt} \iint_S \mathbf{B} \, dS = -\frac{1}{c} \iint S \left\{ \frac{\partial}{\partial t} + (v, \nabla) \right\} \mathbf{B} \, dS, \quad (54)
\]

where \( S \) is any surface bounded by circuit \( C \), moving together with a medium.

This approach is valid only for non-relativistic consideration and leads to Galilean field transformation 43. In Hertz’s theory any motion of the ether relative to the material particles had not been taken into account, so that the moving bodies were regarded simply as homogeneous portions of the medium distinguished only by special values of electric and magnetic constants. Among the consequences of such assumption, Hertz saw the necessity to move the surface of integration in equations (52),(53) at the same time with the moving medium. Thus the generation of a magnetic (or electric) force within a moving dielectric was calculated with implicit use of Galilean invariance in equation (54) unless one makes any additional assumptions on the special character of transformations in a moving frame of reference.

Recently, T. Phipps Jr. again drew attention to the failure of Maxwell’s equations in partial time derivative to describe first-order effects related to convective terms of total time derivatives [41,42]. He proposed to revive Hertz’s Galilean-invariant version of Maxwell’s theory written in total time derivatives. He only differs from Hertz’s own interpretation of the velocity parameter. However, in this review we shall show how total time derivatives can be compatible with the requirements of special relativity in inertial frames of reference.

Let us now examine the case of a point source of electric and magnetic fields. In order to abstain from the use of moving contour \( C \) and surface \( S \) that implies a priori application of some relativity principle (Galileo’s or Einstein’s), we limit our consideration to a fixed region (\( C \) and \( S \) are at rest) whereas the source is moving through a free space. According to Faraday’s law, there must be an electromotive force in the contour \( C \) due to the flux changes with time and convection derivatives simultaneously. Using the mathematical language for total time derivatives, we arrive at the expression analogous to the differential form (47),

\[
\frac{d\Phi}{dt} = \frac{\partial \Phi^*}{\partial t} - (v_x, \nabla)\Phi_0 - (a_x, \nabla)\Phi_0 - \ldots \quad (55)
\]
making use of the definitions:

\[ \Phi^E_0 = \int_S E_0(r - r_s(t))dS, \quad \text{or,} \]
\[ \Phi^B_0 = \int_S B_0(r - r_s(t))dS \]  
\[ \Phi^{(E)} = \int_S E^*(r, t)dS, \quad \text{or,} \]
\[ \Phi^{(B)} = \int_S B^*(r, t)dS, \]  

(56)

where \( r \) is a fixed distance from the point of observation to the origin of the reference systems at rest; \( r_s(t), v_s = dx_s/dt, a_s = dv_s/dt \) are the distance, instant velocity and instant acceleration of the electric (or magnetic) field source.

For the sake of simplicity, we can conserve for the present the same denomination of field flux in two independent parts of total time derivative (56), taking into account additional (fixed space and fixed time) conditions, respectively, in the following expression:

\[ \frac{d}{dt} \Phi = \left\{ \frac{\partial}{\partial t} - (v_s, \nabla_r) - (a_s, \nabla_v) - \ldots \right\} \Phi. \]  
(58)

Using a well-known representation for the convection part in equation (56),

\[ \langle v, \nabla \rangle \int_S E dS = \int_S v(\nabla E)dS + \int_S \left[ \nabla [E, v] \right]dS, \]  
(59)

we obtain an alternative form of Maxwell’s integral equations (52),(53) for a moving electric charge in the reference system at rest,

\[ \int_C \mathbf{H} d\ell = \frac{4\pi}{c} \int_S j dS + \frac{1}{c} \int_S \left\{ \frac{\partial E}{\partial t} - v(\nabla E) - [\nabla, [E, v]] - \ldots \right\} dS, \]  
(60)

\[ \int_C \mathbf{E} d\ell = -\frac{1}{c} \int_S \left\{ \frac{\partial B}{\partial t} + [\nabla, [B, v]] + \ldots \right\} dS. \]  
(61)

Here we omit, for the sake of simplicity, acceleration and other expansion terms in general formula but they, of course, are tacitly implied.

Before going on to a more general consideration of a large number of sources, it is worth to draw attention that we arrived to the most compact differential form of Maxwell- Hertz equations in the reference system at rest [30],

\[ \nabla, \mathbf{E} = 4\pi \rho, \]  
(62)

\[ \nabla, \mathbf{B} = 0, \]  
(63)

\[ [\nabla, \mathbf{H}] = \frac{4\pi}{c} \rho v + \frac{1}{c} \frac{d\mathbf{E}}{dt}, \]  
(64)

\[ [\nabla, \mathbf{H}] = \frac{4\pi}{c} j + \frac{1}{c} \frac{d\mathbf{B}}{dt}, \]  
(65)

where the total time derivative of any vector field value \( \mathbf{E} \) (or \( \mathbf{B} \)) is,

\[ \frac{d\mathbf{E}}{dt} = \frac{\partial \mathbf{E}}{\partial t} - (v, \nabla)\mathbf{E} - (a, \nabla_v)\mathbf{E} - \ldots \]  
(66)

The above-mentioned form (62)–(65) was for the first time admitted by Hertz for electrodynamics of bodies in motion [38,39]. It was the covering theory for Maxwell’s original approach which became the limit case of motionless medium (a reference system at rest) when values of instant velocity \( v \), instant acceleration \( a \) etc. tend to zero in (66) leaving only partial time derivatives in agreement with (6)–(9). The difference of the present approach [30] with Hertz’s covering theory (and with Phipps’ neo-Hertzian approach [41, 42]) consists in the definition of the total time derivative (66) for a medium at rest (not in motion with the possible implication of Galilean invariance). Below we shall demonstrate that the set (62)–(65) possesses invariance properties in any inertial frame of reference.

There is no difficulty in extending this approach to a many particle system, assuming the validity of the electrodynamics superposition principle. This extension is important in order to find out whether the generalised microscopic field equations cover the original (macroscopic) Maxwell’s theory as a limiting case. To do so one ought to take into account all principal restrictions of Maxwell’s equations (6)–(9) which deal only with a continuous and closed (or going off to infinity) conduction currents. They also have to be motionless as a whole (static tubes of charge flow), admitting only the variation of current intensity.

Under these assumptions, it is quite easy to show that the total (macroscopic) convection and others displacement currents are cancelled by itself by summing up all microscopic contributions,

\[ \sum_i (v_i, \nabla)\mathbf{E}_i + (a_i, \nabla_v)\mathbf{E}_i + \ldots = 0, \]  
(67)

\[ \sum_i (v_i, \nabla)\mathbf{B}_i + (a_i, \nabla_v)\mathbf{B}_i + \ldots = 0. \]  

In other words, every additional terms in (50),(51) (as well as in (60),(61) disappears and we obtain the original set of Maxwell macroscopic equations (6)–(9) for continuous and closed (or going off to infinity) conduction currents as a valid approximation.

To conclude this part we would like to note that the set of equations (60),(61) can be called as
modified Maxwell-Hertz’s equations extended to one charge system. It is easy to see that in this form they are completely equivalent to modified Maxwell-Lorentz equations (50),(51) obtained with the help of the balance equation. Thus, differential and integral approaches to extend the original Maxwell theory lead to the same result.

Let us write once again the generalised form of Maxwell-Lorentz equations explicitly for a single moving particle that is a source of electric and magnetic fields simultaneously,

\begin{align}
(\nabla, \mathbf{E}) &= 4\pi \rho, \\
(\nabla, \mathbf{H}) &= 0,
\end{align}

where

\begin{align}
[\nabla, \mathbf{H}] &= \frac{4\pi}{c} \rho \mathbf{v} + \frac{1}{c} \left( \frac{\partial \mathbf{E}^*}{\partial t} - (\mathbf{v}, \nabla) \mathbf{E}_0 - (\mathbf{a}, \nabla_v) \mathbf{E}_0 - \ldots \right), \\
[\nabla, \mathbf{E}] &= -\frac{1}{c} \frac{\partial \mathbf{B}^*}{\partial t} - \frac{1}{c} \left[ \nabla, [\mathbf{v}, \mathbf{B}] - \frac{1}{c} (\mathbf{a}, \nabla_v) \mathbf{B}_0 - \ldots \right],
\end{align}

at the same time with the balance equation,

\begin{equation}
d\rho + (\nabla, \rho \mathbf{v}) = 0. \tag{72}
\end{equation}

Splitting up field components into explicit and implicit time-dependent contributions \( \mathbf{E}^* \) (\( \mathbf{B}^* \)) and \( \mathbf{E}_0 \) (\( \mathbf{B}_0 \)), respectively, the basic field equations (70),(71) can be rewritten as follows:

\begin{align}
[\nabla, \mathbf{H}] &= \frac{4\pi}{c} \rho \mathbf{v} + \frac{1}{c} \left( \frac{\partial \mathbf{E}^*}{\partial t} - (\mathbf{v}, \nabla) \mathbf{E}_0 - (\mathbf{a}, \nabla_v) \mathbf{E}_0 - \ldots \right), \\
[\nabla, \mathbf{E}] &= -\frac{1}{c} \frac{\partial \mathbf{B}^*}{\partial t} - \frac{1}{c} \left[ \nabla, [\mathbf{v}, \mathbf{B}] - \frac{1}{c} (\mathbf{a}, \nabla_v) \mathbf{B}_0 - \ldots \right], \tag{73}
\end{align}

where the total field values have two independent parts,

\begin{align}
\mathbf{E} &= \mathbf{E}_0 + \mathbf{E}^* = \mathbf{E}_0 (\mathbf{r} - \mathbf{r}_0(t)) + \mathbf{E}^* (\mathbf{r}, t), \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}^* = \mathbf{B}_0 (\mathbf{r} - \mathbf{r}_0(t)) + \mathbf{B}^* (\mathbf{r}, t). \tag{74}
\end{align}

Here we note that implicit time-dependent field components \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \) depend only on the point of observation and on the source position at an instant whereas time varying-fields \( \mathbf{E}^* \) and \( \mathbf{B}^* \) depend explicitly on time at a fixed point. The separation procedure may be similarly extended to the electric and magnetic potentials introduced as

\begin{align}
\mathbf{E} &= \nabla \varphi, \\
\mathbf{B} &= [\nabla, \mathbf{A}], \tag{77}
\end{align}

where

\begin{align}
\varphi &= \varphi_0 + \varphi^*, \\
\mathbf{A} &= \mathbf{A}_0 + \mathbf{A}^*. \tag{78}
\end{align}

Let us establish invariance of field equations in total time derivatives. As far as in special relativity the invariance is looking for inertial frames of reference moving with a constant velocity \( \mathbf{v} \), then in total time derivative expansion we should omit all acceleration and higher order terms. Thus, using definitions (77),(78) we obtain from equation (74) that

\begin{align}
\mathbf{E} &= -\nabla \varphi - \frac{1}{c} \frac{\partial \mathbf{A}^*}{\partial t} - \frac{1}{c} \mathbf{v}[\mathbf{v}, \mathbf{B}_0]. \tag{79}
\end{align}

Separation of implicit time-dependent from explicit time-dependent components in (79) is straightforward

\begin{align}
\mathbf{E}_0 &= -\nabla \varphi_0 - \frac{1}{c} \mathbf{v}[\mathbf{v}, \mathbf{B}_0], \\
\mathbf{E}^* &= -\nabla \varphi^* - \frac{1}{c} \frac{\partial \mathbf{A}^*}{\partial t}.
\end{align}

Using this separation we obtain two second order differential equations for total potentials (78)

\begin{align}
\Delta \mathbf{A} &= -\frac{4\pi}{c} \rho \mathbf{v} + \mathbf{F}_1, \tag{81}
\Delta \varphi &= -4\pi \rho + \mathbf{F}_2, \tag{82}
\end{align}

where

\begin{align}
\mathbf{F}_1 &= \nabla \left( \nabla, \mathbf{A}_0 + \mathbf{A}^* \right) - \frac{1}{c} (\mathbf{v}, \nabla) \nabla \varphi_0 + \frac{1}{c} \frac{\partial \nabla \varphi^*}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}^*}{\partial t^2}, \tag{83}
\mathbf{F}_2 &= -\frac{1}{c} \left( \nabla, \frac{\partial \mathbf{A}^*}{\partial t} \right), \tag{84}
\end{align}

The second term in (83) can be easily transformed using mathematical operations of field theory,

\begin{equation}
(\mathbf{v}, \nabla) \nabla \varphi_0 = \nabla (\mathbf{v}, \nabla \varphi_0) - [\mathbf{v}, [\nabla, \nabla \varphi_0]]. \tag{85}
\end{equation}

Since \( [\nabla, \nabla(\ldots)] \) is always equal to zero, we can rewrite \( \mathbf{F}_1 \) in a new form,

\begin{align}
\mathbf{F}_1 &= \nabla \left[ \left( \nabla, \mathbf{A}_0 \right) - \frac{1}{c} (\mathbf{v}, \nabla \varphi_0) \right] + \\
&\quad \left[ [\nabla, \mathbf{A}^*] + \frac{1}{c} \frac{\partial \varphi^*}{\partial t} \right] + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}^*}{\partial t^2}. \tag{86}
\end{align}

The principal feature of (86) consists in the fact that all implicit and explicit time-dependent components of total electric and magnetic potentials enter independently and, therefore, can be characterized by respective gauge conditions,

\begin{align}
\nabla \cdot \mathbf{A}_0 - \frac{1}{c} (\mathbf{v}, \nabla \varphi_0) &= 0, \tag{87}
\nabla \cdot \mathbf{A}^* + \frac{1}{c} \frac{\partial \varphi^*}{\partial t} &= 0. \tag{88}
\end{align}

Lorentz’s gauge (88) is applicable now only for explicit time-dependent potentials and is invariant under
Lorentz’s transformations. It suggests that the proper time \( \tau \) (let us call here \( \tau \) the time variable of explicit time-dependent components in the entire spirit of the special relativity theory) for two inertial frames moving with respect to each other are related by an imaginary rotation in space-time. The amount of rotation depends on the relative velocity.

Implicit time-dependent potentials turn out to be related through the novel gauge (87) which covers a well-known relationship between the components of electric and magnetic field potentials of uniformly moving charge [33],

\[
A_0 = \frac{v}{c} \varphi_0. \tag{89}
\]

Strictly speaking, this relationship is true for Galilean as well as for Lorentz’s transformations. The difference is attributed to a mathematical formulation of potentials in a new frame of reference. For instance, the Lorentz transformation corresponds to a rotation in the space-time plane whereas the Galilean one leaves \( A_0 \) and \( \varphi_0 \) unchanged, for it is assumed that no operation can rotate the time axis into the space axis or vice versa. For Galilean invariance, the time direction is supposed to be the same for all inertial frames of reference.

The expression (87) and all physically possible transformations based on it, do not involve explicitly any time dimension. The time \( t \) here can be added as an afterthought (a parameter describing the space coordinate \( R(t) \)). In above discussion of full time derivative we noted that time variable \( \tau \) (for explicit) and time parameter \( t \) (for implicit time behaviours) are, generally speaking, independent. If we assume, as they do it tacitly in special relativity with no distinction of time behaviours, that both time variables are identical \( t = \tau \) then we arrive to the implication of Lorentz’s invariance for \( A_0 \) and \( \varphi_0 \). Without additional hypothesis, the present Helmholtzian approach cannot rule in favour of Galilean or Lorentz’s transformations for implicit time dependences. The novel gauge (87) as well as (89) are compatible with both of them. The only way of resolving this dilemma now seems to be to suggest experimental verification of electric field transformation in a moving frame. In fact, Leus recently proposed such experiment [51]. A uniform beam of electrons moving with the velocity close to \( c \) has to produce electric field strenght which differs for Galilean and Lorentz transformations.

Two gauge conditions (87) and (88) can be written jointly in a more compact formula that we can call the generalized Lorentz condition,

\[
\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0, \tag{90}
\]

where \( A_0 \) and \( \varphi_0 \) are defined by (78) and the total time derivative is taken as in (66) up to the convection term.

We have not done it yet and will do it elsewhere but, perhaps, it is possible to prove that generalised Lorentz gauge (90) is valid also for non-inertial frames (acceleration and higher order terms in the total time derivative expansion). It would have a very attractive consequence that the field equations (62)-(65) written in total time derivatives could be considered invariant regardless a frame of reference (inertial or non-inertial).

Recall that in special relativity, electric and magnetic potentials of uniformly moving charge \( A_0 \) and \( \varphi_0 \) are interrelated through the relationship (89) under application of Lorentz transformation. Here we found that relativistic potentials (or components of potential four-vector) are connected in a more general way (87). Another important aspect of the present approach can be attributed to the verification of some ambiguity in the use of Lorentz gauge since it is applicable only to explicit time-dependent potentials. In fact, there are some difficulties in the conventional electrodynamics concerning the inconsistency of this gauge with implicit time-dependent functions. The standard Lorentz gauge condition

\[
\nabla \cdot \mathbf{A} = 0. \tag{91}
\]

is assumed to be valid for total electric and magnetic potentials (transverse plus longitudinal) and is considered suffice to hold Maxwell’s equations invariant under Lorentz transformation. In the quasi-stationary approximation, the Lorentz condition in every frame of references takes the form of the so-called radiation gauge [43],

\[
\nabla \cdot \mathbf{A} = 0, \tag{92}
\]

It contradicts the expected relation (89) (or in our approach (87)) between electric and magnetic implicit time-dependent potentials. To make (92) consistent with (89) in the given frame, they used to put an additional condition on the electric potential satisfying the so-called Coulomb gauge [43],

\[
\nabla \cdot \mathbf{A} = 0, \quad \varphi = 0. \tag{93}
\]

In mathematical language the invariance of implicit time-dependent fields in the conventional approach involves more strong limitations than those imposed previously by the Lorentz gauge. Generally speaking, the conventional classical electrodynamics has to admit more than one invariance principle since every time the Lorentz transformation is done, one needs also simultaneously to transform all physical quantities in accordance with the Coulomb gauge (93). This problem was widely discussed and in the language adopted in the general Lorentz group theory, is known as gauge dependent representation (or joint representation) of the Lorentz group [43]. In fact, it means an additional non-relativistic adjustment of
electric potential, every time we change the frame of reference. This difficulty vanishes when the relativistic gauge (87) for implicit time-dependent potentials is introduced.

A rigorous consideration of (81), (82) gives another important conclusion: simultaneous application of two independent gauge transformations (87), (88) decomposes the initial set (68)–(71) into two pairs of differential equations, namely,

\[ \Delta A_0 = -\frac{4\pi}{c}\rho \mathbf{v}, \quad (94) \]
\[ \Delta \varphi_0 = -4\pi \rho \quad (95) \]

at the same time with the homogeneous wave equations,

\[ \Delta A^* = -\frac{4\pi}{c} \frac{\partial^2 A^*}{\partial t^2} = 0, \quad (96) \]
\[ \Delta \varphi^* = -\frac{4\pi}{c} \frac{\partial^2 \varphi^*}{\partial t^2} = 0. \quad (97) \]

Likewise (89), Poisson’s second order differential equations (94), (95) for electric and magnetic potentials covers the conventional approach in the steady-state approximation and can be considered as valid extension to implicit time-dependent potentials. A general solution, as one would expect, satisfies a pair of uncoupled inhomogeneous D’Alembert’s equations. It can be verified by summing up (94), (95) and (96), (97) (here we omit premeditatedly all boundary conditions for the sake of simplicity),

\[ \Delta A = -\frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} \rho \mathbf{v}, \quad (98) \]
\[ \Delta \varphi = -\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -4\pi \rho, \quad (99) \]

where the total values \( A \) and \( \varphi \) are defined by (78).

The same result has been derived in the Section VI independently, starting from the analysis of boundary conditions for inhomogeneous D’Alembert’s equations [29]. It has been shown mathematically that any general solution of Maxwell’s equations has to be obligatory written as a superposition of implicit and explicit time-dependent functions. The above analysis endorsed that conclusion by demonstrating relativistic invariance of (98), (99) and, therefore, (68)–(71), if and only if the relativistic gauge condition (90) is satisfied by respective components of the total field. Thus, the covering theory based on the total time derivatives possesses all necessary relativistic symmetry properties.

To conclude this section, some remarks worth to be done concerning the empirical and axiomatic status of the Lorentz force concept in the electron theory formulated by Lorentz. In the first version of Maxwell’s theory published under the name “On Physical Lines of Force” (1861–1862) there was already admitted an unified character of a full electromotive force in the conductor in motion by describing it as \([44, 45]\)

\[ \mathbf{E} = -\nabla \varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c} [\mathbf{v}, \mathbf{B}], \quad (100) \]

where (1) the first term is the electrostatic force, (2) the second is the force of magnetic induction and (3) the third one is the force of electromagnetic induction due to the conductor motion. Later investigations began to distinguish between the electric force in a moving body and the electric force in the ether through which the body was moving and as a result, did not consider \(1/c[\mathbf{v}, \mathbf{B}]\) as a full-value part of the electric field, as afterwards was argued by Hertz. This distinction was one of the basic premises in Lorentz’s electron theory and was closely related to the special status of the Lorentz force conception. It also can be noted in the way how it forms part the formalism of the conventional field theory. The equation of motion with total time derivative (30) should be contrasted from the form of partial differential equations (26)–(29). It does not correspond to the mathematical structure of a consistent system.

In special relativity the Lorentz force, is the result of the transformation of the components of Minkowski’s force. Thus, the expression for the Lorentz force can be obtained in a purely mathematical way from the general relativistic relationships [33]. In the present Helmholtz-type approach the Lorentz force is one of the terms in the total time derivative expansion. This has advantage to be consistent by itself with the set of generalised field equations. There is no need to supplement Maxwell’s theory with equation of motion. Given such interpretation of Lorentz’s force, we remind that in our approach it can be related only to implicit time-dependent components whereas in the conventional electrodynamics it was the product of the total magnetic field leading to some ambiguities. In this respect it is interesting to mention very recent works by Wesley [46] and Phipps [47] challenging the sufficiency of the Lorentz force law to describe experimental observations. They advocated the use of total time derivatives (in the above-mentioned neo-Hertzian sense) and their data roughly agreed with theoretical predictions, while the conventional theory does not predict any effect at all.

9. Analysis of classical difficulties and the Hamiltonian form of generalized Maxwell’s equations

Maxwell’s equation in the form of D’Alembert’s equations lend themselves to the covariant description and are in agreement with the requirements of special
relativistic mathematical formalism. For four-vectors of separated potentials, the standard four-vector form of basic equations can be used. We immediately have the following expressions:

$$\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) [A_{0\mu} + A^\ast_{\mu}] = - \frac{4\pi}{c} J_\mu, \quad \mu = 0, 1, 2, 3, \quad (101)$$

where

$$A_{0\mu} + A^\ast_{\mu} = (\varphi_0 + \varphi^*, A_0 + A^*), \quad J_\mu = (e\rho, j). \quad (102)$$

The first Poisson’s operator $\Delta$ acts only on the four-vector of implicit time-dependent components $A_{0\mu}$ whereas $\Delta$ and $\partial^2/\partial t^2$ act together on explicit time-dependent components $A^\ast_{\mu}$. The equation (101) is relativistically invariant under the generalized relativistic Lorentz gauge condition (90). To give some substance to the above formalism we exhibit explicitly Poisson’s equation for implicit time-dependent four-vector $A_{0\mu}$

$$\Delta A_{0\mu} = - \frac{4\pi}{c} J_\mu, \quad (103)$$

where

$$A_{0\mu} = (\varphi_0, A_0). \quad (104)$$

As we demonstrated in the previous Section, equation (103) is relativistically invariant under the Lorentz gauge (83) if the time parameter $t$ here is considered identical to the time variable $\tau$ for explicit time components $A^\ast_{\mu}$. Under this condition, Poisson’s differential operator $\Delta$ acting on implicit time-dependent potentials becomes invariant in every inertial frame of reference under Lorentz’s transformations. This is due to the fact that time variable $t$ is not any more independent from $\tau$ as it is assumed for partial derivatives in full time derivative formalism. Non-covariant representation of D’Alembert differential operator $\Delta - \partial^2/\partial t^2$ or, in other words, non-covariance of equation (103) is not a stumbling block here for relativistic invariance and endorses the well-known fact that covariance is not necessary, it is only sufficient for relativistic invariance.

More over, it is tacitly implied in the conventional approach and corresponds to the relativistic invariance of field components of an uniformly moving charge (implicit time-dependent functions) that remain radial lines of electric field regardless the choice of inertial frame. This fact is odd to contemplate in the Faraday-Maxwell electrodynamics based on the concept of local (contact) field which mathematically fits explicit time-dependent behaviour.

Actually, electric field lines of an unmoving charge are radial. Under Lorentz’s transformation into the inertial frame of reference moving with the velocity $v$ explicit time-dependence does not appear and field lines remain radial. Without any approximation, the influence of a possible retarded effect cancels itself at any distance from the moving charge.

On the other hand, the conventional theory is unable to give any reasonable interpretation describing a transition from an uniform movement of a charge into an arbitrary one and then again into uniform over a limited interval of time. In this case, the first and the latter solutions can be given exactly by the Lorentz transformation as implicit time-dependent functions. What mechanism changes them at a distance unreachable for retarded Lienard-Wiechert fields? The lack of continuity between the corresponding solutions is obvious. It has the same nature as discussed in the Section VI.

The Helmholtz-type approach based on separation of implicit and explicit time behaviours, also highlights serious ambiguities associated with the self-energy concept in the framework of the conventional electrodynamics. Let us confine our previous qualitative reasoning to the example of electrostatics. A rigorous analysis will be done later applying Hamiltonian formalism.

In electrostatics the total energy of $N$ interacting charges is

$$W = \frac{1}{2} \sum_{i=1}^{N} \sum_{j\neq1} \frac{e_ie_j}{|r_i - r_j|}. \quad (105)$$

Here, the infinite self-energy terms ($i = j$) are omitted in the double sum. The expression obtained by Maxwell for the energy in an electric field, expressed as a volume integral over the field, is [45]

$$W = \frac{1}{2} \int_V E^2 dV \quad (106)$$

This corresponds to Maxwell’s idea that the system energy must be stored somewhere in space. The expression (106) includes self-energy terms and in the case of point charges they make infinite contributions to the integral.

In a relativistically covariant formulation the conservation of energy and the conservation of momentum are not independent principles. In particular, the local form of energy-momentum conservation can be written in a covariant form, using the energy-momentum tensor,

$$\frac{\partial T^\mu{}_{\nu}}{\partial x^\mu} = 0. \quad (107)$$

For an electromagnetic field, it is well-known that (107) can be strictly satisfied only for a free field (when a charge is not taken into account), whereas, for the total field of a charge this is not true, since (107) is not satisfied mathematically (four-dimensional analogy of Gauss’s theorem). As everyone knows in classical electrodynamics, this fact gives
rise to the "electromagnetic mass" concept, which violates the exact relativistic mass-energy relationship \( E = mc^2 \). Let us examine this problem in a less formal manner. The equivalent three-dimensional form of (107) is the formula (3). The amount of electrostatic self-energy of an unmoving charge in a given volume \( V \) is proportional to \( E^2 \) (see (106)). According to (107) (or (105)), in a new inertial frame, energy density \( W \) as well as electric field \( E \) must be, generally speaking, an explicit time-dependent function \( \left( \partial W / \partial t \neq 0 \right. \) and \( \partial E / \partial t \neq 0 \). On the other hand, the electric field strength of an unmoving charge keeps its implicit time behaviour under Lorentz's transformation \( \left( \partial E / \partial t = 0 \right) \). It contradicts the commonly accepted view that electrostatic self-energy is stored locally in space.

In the framework of Helmholtzian approach these ambiguities can be cleared up. Actually, looking back at the general solution (23) with explicitly exposed longitudinal and transverse components, the term \( E_0 \) is responsible for bipartite interaction between charges. No local energy conservation law in the form (107) or (3) is adequate for implicit time-dependent field \( E_0 \). We suggest that the original mathematical form (105) should be used. Nevertheless, the local form (107) or (3) is perfectly adequate for explicitly time-dependent free field \( E' \). Clear separation on implicit and explicit time dependencies in Helmhotz-type electrodynamics leads to the correspondent separation in the total electric field energy expression,

\[
W = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{e_i e_j}{|r_i - r_j|} + \frac{1}{2} \int E^2 dV. \tag{108}
\]

This is a logical conclusion of our qualitative reasoning that will be mathematically verified below in Hamiltonian formulation.

Let us discuss generalised field equations in total time derivatives (62)–(65) for arbitrary fields from the standpoint of the principle of least action. Applying explicitly separation of field components we have not done any modifications in the general four-vector representation of Maxwell equations (101), (102). We only noted that in this case the set of field equations can be split up for equations of implicit and explicit time-dependent potentials such as (16)–(19) or (94)–(97). A relativistic action for implicit time potential \( A_{0\mu} \) can be written in the conventional form [33],

\[
S_m + S_{mf} = \int \left( \frac{1}{4} \sum_{a=1}^{N} m_a c \, ds_a - \sum_{a=1}^{N} \sum_{\mu=0}^{3} A_{0(a\mu)} \, dx_a^\mu \right). \tag{109}
\]

This expression is sufficient to derive the first couple of equations (16), (17) (or (94), (95)) from the least action principle. It can be directly verified by rewriting the second term in (109) as

\[
S_{mf} = -\frac{1}{c} \int \sum_{\mu} A_{0\mu} j^\mu dV dt \tag{110}
\]

and using Dirac's expression for four-current,

\[
J_\mu(r, t) = \sum_a \left[ -\frac{e_a}{4\pi} \Delta \left( \frac{a}{r - r_a} \right) \right] U_{\mu a}. \tag{111}
\]

where \( U_{\mu a} \) is the four-velocity of the charged particle \( a \).

Let us consider the second pair of equations (18), (19) or (96), (97) defining explicitly time-dependent potentials \( (\varphi^*, A^*) \) or \( A^\mu_\nu \) in representation (101). It is easy to see that the conventional Hamiltonian form can be adopted to describe transverse components of electromagnetic field [33],

\[
S_t = -\frac{1}{16\pi} \int \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} dV dt, \tag{112}
\]

where

\[
F_{\mu\nu} = \frac{\partial A^\nu_\mu}{\partial x^\mu} - \frac{\partial A^\mu_\nu}{\partial x^\nu}. \tag{113}
\]

Finally, it remains to be proved that the variational derivative,

\[
\delta S_f = -\int \sum_{\mu} \left( \frac{1}{4\pi} \sum_{\nu} \frac{\partial F_{\mu\nu}}{\partial x^\nu} \right) \delta A^\nu_\mu dV dt \tag{114}
\]

can be used to obtain the covariant analogue of (18), (19) (or (95)–(97)) in the following form:

\[
\sum_{\nu} \frac{\partial}{\partial x^\nu} F^{\mu\nu} = \sum_{\nu} \frac{\partial}{\partial x^\nu} \left[ \frac{\partial A^{\nu\mu}}{\partial x^\mu} - \frac{\partial A^{\mu\nu}}{\partial x^\nu} \right] = 0. \tag{115}
\]

The difference with the conventional interpretation consists in the way electromagnetic potentials \( A_{0\mu} \) and \( A^\mu_\nu \) take part in this Hamiltonian formulation. In the light of the Helmholtzian approach, the electromagnetic energy-momentum tensor demands some corrections in the interpretation of its mathematical formulation [33],

\[
T^{\mu\nu} = -\frac{1}{4\pi} \sum_p F_p^{\nu\rho} F^{\rho\mu} + \frac{1}{16\pi} g^{\mu\nu} \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma}. \tag{116}
\]

As a consequence of the definition (113), it can describe the energy-momentum conservation law for, exclusively, free electromagnetic field as follows,

\[
\sum_{\nu} \frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0. \tag{117}
\]

Consequently, contrary to the traditional interpretation, the quantity \( F^{\mu\nu} \) can be defined
as a transverse electromagnetic field tensor because it contains only transverse field components but not total as in the conventional approach. There is no more violation of (117) even if the charge is taken into account, contrary to the situation in the conventional theory (see above discussion of equation (107)).

Strictly speaking, the total field energy $W$ should be split up into two parts:

1. energy $W_{mf}$ longitudinal implicit time-dependent fields responsible for electro- and magnetostatic interaction between charges (non-local term) and

2. energy $W_f$ of transverse explicitly time-dependent electromagnetic field (local term),

$$W = W_{mf} + W_f.$$  \hfill (118)

Following these results we suggest that the concept of potential (non-local) energy and potential forces must be re-established in classical electrodynamics. So, the system of charges and currents in absence of free electromagnetic field $W_f$ must be considered as a conservative system without any idealization. Introduction of interaction energy $W_{mf}$ in the form (109) equivalent to (105) definitely eliminates the problem of infinities of self-energy terms.

The physical meaning of the Poynting vector has been changed notably. So far the conventional theory dealt with it as a quantity describing dynamic properties of the total electromagnetic field. Now it is adequate only for conservation law in the form of equation (117) and, therefore, makes sense only for transverse components of electromagnetic field. Long-time well-known ambiguities related to the definition of the field energy location in space, do not take place in Helmholtz-type electrodynamics. In particular, there should be no flux of electromagnetic energy for stationary currents. Contrary, the conventional approach predicts senseless flux of energy coming from infinity towards the current [28].

At the end of this Section we would like to present a valuable mechanical analogy of Maxwell’s equations in the form of (16)–(19) (or 94)–(97). It helps to understand why general solutions must be considered as mechanical analogy of Maxwell’s equations respectively. The set of differential equations for elastic waves in an isotropic media (see [48]) can present a valuable mechanical analogy of Maxwell’s solution. Generally, the general solution of (110),(111) is the sum of two independent and orthogonal terms corresponding to longitudinal $u_l$ and transversal $u_t$ waves,

$$u = u_l + u_t.$$  \hfill (121)

If the longitudinal spreading velocity approaches formally to infinity ($c_l \to \infty$) then (119) transforms into Laplace’s equation whereas the general solution turns out to have an implicit time dependence. Solution (121) takes the form of separated potential solution (14),(15) (or (78)). Longitudinal component does not vanish in this limit from mathematical consideration, though the time behaviour undergoes a fundamental transformation. Thus, longitudinal waves $u_l$ have to be considered as full-value solution of the total system of differential equations (119),(120). It allows to understand why Hertz had no right to eliminate longitudinal components from mathematical solutions of Helmholtz’s theory in Maxwellian limit (see corresponding discussion in the Section II, quote [21]).

To end the Section we conclude that the idea of non-local interactions is enclosed into the framework of Helmholtzian electromagnetic theory as unambiguous mathematical feature. On the other hand, some of the quantum mechanical effects like Aharonov-Bohm effect, violation of the Bell’s inequalities etc. point out indirectly on the possibility of non-local interactions in electromagnetism. During the last century modern physics had faced fundamental difficulties in unifying relativistic classical physics elaborated mainly in the framework of the locality concept of relativistic theory and quantum physics characterized essentially by the emergence of non-locality. Regrettfully, nowadays there is no rigorous mutual correspondence between these two fundamental areas of physical science. Helmholtz-type approach offers an altogether more promising solution.

10. Non-radiation condition for free electromagnetic field

In this section we make a qualitative discussion of the energy balance between the system of interacting charged particles and free electromagnetic field, namely, energy and momentum lost by radiation. We must examine carefully one essential difference in the electromagnetic energy interpretation. Let us write the total relativistic action as

$$S = S_m + S_{mf} + S_f.$$  \hfill (122)

Although we adopt denominations used in the conventional theory, the physical essence of the last two terms has changed significantly. Usually, the interaction between particles and electromagnetic
field was attributed to $S_{mf}$ whereas the properties of electromagnetic field manifested itself by the additional term $S_f$.

In the new approach, no concept of local (contact) field as intermediary is suffice to describe the interaction between charges (currents). Hence, $S_{mf}$ is understood in terms of non-local conservative field. Local field (transverse components attributed to radiation or free field) is represented by the last term $S_f$. The possible free field interaction with the system of charges (currents) depends entirely on its location in space.

The internal structure of the relativistic action (122), where local and non-local contributions are separated in different terms, allow us to consider the isolated system of charged particles and free field as consisting of two corresponding subsystems. Each of the subsystems may be completely independent if there is no mutual interaction (for instance, free electromagnetic field is located far from the given region of charges and currents). In static and steady approximations the subsystem of charges and currents can be considered as conservative. The total Hamiltonian of our isolated system can be split up into two corresponding parts,

$$H = H_1 + H_2,$$  \hspace{1cm} (123)

where $H_1$ is the Hamiltonian of the conservative system of charges and currents. It involves apart from electro- and magnetostatic energy also mechanical energy of particles (corresponding to the action $S_m + S_{mf}$). $H_2$ is the Hamiltonian of the free electromagnetic field (corresponds to the action $S_f$).

It is important to note that the separation into two subsystems is possible only in this new approach. In the conventional interpretation of $S_f$ described the properties of the total electromagnetic field without making any distinction between longitudinal and transverse components. In static approximation the term $S_f$ should vanish in our approach but in the conventional approach it is not zero, and corresponds to the field self-energy [33],

$$S_f = \int_{t_1}^{t_2} L_f dt,$$  \hspace{1cm} (124)

where

$$L_f = \frac{1}{8\pi} \int_V (E^2 - B^2) dV.$$  \hspace{1cm} (125)

Here $E$ and $B$ are the total electric and magnetic field strengths, respectively, admitted in the conventional approach.

In relativistic theory, energy is zero component of the four-momentum. For an isolated system the total Hamiltonian is not time-dependent and, therefore, the energy conservation as well as the momentum conservation may be treated independently. Conservation laws are useful tools to get some general insight on the problem even when complete solutions are very difficult to find.

Let us consider the case where charges (currents) and free electromagnetic field are located in the same region and become interacting. Internal forces of mutual interaction between two subsystems are usually named as internal dissipative forces. They carry out the energy exchange inside the total isolated system. In terms of the Hamiltonian formalism [49] it can be expressed as a corresponding Hamiltonian evolution,

$$\frac{dH_{1,2}}{dt} = \frac{\partial H_{1,2}}{\partial t} + P_{1,2}^{ex} + P_{1,2}^{in},$$  \hspace{1cm} (126)

where $P_{1,2}^{ex}$ ($P_{1,2}^{in}$) is the power of the external (internal) forces acting on two subsystems, respectively. In our case $P_{1}^{ex}$ and $P_{2}^{ex}$ appear as a result of mutual interaction. On the other hand, any internal non-potential force in the first subsystem can also cause energy dissipation ($P_{1}^{in}$). Even in the absence of a real mechanical friction, other internal non-potentials forces (for example, inhomogeneous gyroscopic forces) can still act in this subsystem and dissipate energy. In other words, if initially there is no free electromagnetic field ($H_2 = 0$), it can be created by internal non-potential forces ($P_{1}^{in}$) acting in the first subsystem ($H_2$ is no more zero). It means that energy is lost by radiation in the subsystem of charges and currents.

In mathematical language the corresponding energy balance can be written as follows:

$$\frac{d}{dt}(H_1 + H_2) = 0$$  \hspace{1cm} (127)

where $\frac{d}{dt}(H_1)$ and $\frac{d}{dt}(H_2)$ are energy change rates for the first and the second subsystems, respectively. It might be easily noted that the energy balance (127) is symmetrical with respect to time reversion. This feature is in accordance with the time symmetry of generalized Maxwell’s equations in total time derivatives.

Here it should be specially noted that the energy balance in the conventional electrodynamics is not always reversible in time. For instance, the dipolar radiation energy is proportional to the square of a charge acceleration. Thus, the total dipolar radiation energy is always positive regardless the mathematical operation on the reversion of time (negative sign of time variable). It indicates (not proves) that conventional Maxwell’s equations may not be entirely time symmetrical. Perhaps, the failure to build up classical Rutherford’s model of the atom is due to that deficiency.

To end this Section we formulate the previous statement about the energy conservation as the condition of non-radiation of the free electromagnetic field: \textit{if in an isolated system of charges (currents) in the absence of free electromagnetic field ($d/dt(H_1) = 0$), all internal non-potential forces are compensated
or do not exist then this system will not produce (radiate) free electromagnetic field (remains zero $H_2 = 0$) and will keep the conservative system itself.

This implies not only an equilibrium between radiation and absorption but no radiation at all. This possibility would be of particular interest in the attempt to understand the quantum mechanical principles and may be explored elsewhere.

11. Conclusions

From the modern philosophy of science standpoint, then, it is plain that Helmholtz's theory could not be definitely ruled out, since it was perfectly falsifiable by Popper’s criterion, did not contradict any observable fact and predicted all well-known electromagnetic phenomena in Maxwellian limit. As the relevant history of physics literature shows, Hertz himself indicated in his last experimental work that he definitely could not disprove any reference to action at a distance. Moreover, Hertz’s crucial experiments provided no explicit information on longitudinal components which were such an essential feature of Helmholtz’s theory. From this point they could not be considered conclusive to refute alternative approach. This may indicate that there was a need for further experimental investigations of Hertz into the possibility that Helmholtz’s theoretically predicted longitudinal electromagnetic components did in fact exist. More explicit and direct experimental information on the existence of longitudinal forces should have determined the Hertz choice between Maxwell’s and Helmholtz’s theories. As far as this experimental part was not fulfilled, the problem of the completeness of Hertz’s experiments on propagation of electromagnetic interactions could not be considered as fully resolved. Thus, according to the modern criteria of scientific method, philosophy and methodology of science say that Hertz’s experiments cannot be considered as conclusive at some points as it is generally implied.

Mathematical analysis of Maxwell-Lorentz equations for one charge system shows ambiguous conventional treatment of implicit and explicit time dependencies. It was found that all conventional approach is beset with the same ambiguity leading to many mathematical inconsistencies and paradoxes. We suggested that it is possible to solve those difficulties by clear distinguishing between functions with implicit and explicit time dependencies. This consideration provided self-consistency for mathematical description of electromagnetic theory. Maxwell’s equations resulted to be written in full time derivatives that consistently covers conventional approach. We showed that the covering theory possesses all necessary relativistic invariance properties for inertial frames of references. Usual Lorentz’s gauge condition is covered by generalised gauge condition. It promises to keep generalised Maxwell’s equations invariant also in non-inertial frames but this issue will be studied elsewhere.

Consistent mathematical interpretation of generalised field equations gives a solid ground for Helmholtzian foundations of classical electrodynamics based on the superposition of implicit time dependent longitudinal and explicit time dependent transverse components. This approach demonstrates advantages over the conventional field description in eliminating the large number of internal inconsistencies from classical electrodynamics and promises more adequate solution to fundamental problems of modern physics. Recent experimental data [46, 47] highlighted certain limitations of the conventional approach. Graneau’s monograph on modern Newtonian electrodynamics [50] reviewed numerous research data in exploding wires, railguns, different electromagnetic accelerators, jet propulsion in liquid metals, plasma explosions, capillary fusion etc. as unambiguous indication on the existence of non-local longitudinal forces. Thus, a new area of electromagnetic research emerges that is interested in the study of longitudinal components by experimental as well as by theoretical means.

As a final conclusion of this review, we would like to quote P. Duhem’s significant words [27]: “... An excessive admiration for Maxwell’s work has led many physicists to the view that it does not matter whether a theory is logical or absurd, all it is required to do is suggest experiments: A day will come, I am certain, when it will be recognised: that above all the objects of a theory is to bring classification and order into the chaos of facts shown by experience. Then it will be acknowledged that Helmholtz’s electrodynamics is a fine work and that I did well to adhere to it. Logic can be patient, for it is eternal”.

References


[18] ibid., p.108

[19] ibid., p.121


[23] ibid., pp. 151-152


[27] Duhem P. Les Théories Electriques de J. Clerk Maxwell – Paris: – 1902 (quoted from Ref. 15)


