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Elasticity and Electromagnetism (The Coulomb Gauge)

Contents

Abstract

The Maxwell's electromagnetic equations are isomorphic to the motion equation of a linear elastic continuum which is hard to compression though liable to shear deformation. The Coulomb gauge expresses the medium incompressibility. The magnetic vector potential corresponds to the medium velocity. The pressure stands for the electrostatic potential. The electric field is modeled by an external force whose origin is beyond the elastic model. The electric charge corresponds to a medium defect which produces the perturbation $\delta p \sim 1/r$ of the pressure field. The defects interact with each other due to the dilatation center in the core of the inclusion.

It is well known that classical electromagnetism is similar in its structure to the theory of the strain and stress of a solid elastic body [1]. Nevertheless, they failed to find such a solid medium, whose motion equation reproduces precisely the Maxwell's equations. In other words, there is no solid medium realization of the electromagnetic substratum. However, introducing respective force terms into the equation of linear elasticity, the structure of the classical electromagnetism equations can be accurately imitated. It will be useful to do this work consistently, thus indicating the position of the mechanical model of electromagnetism among other models of elasticity. Logically, this stage of the construction of the mechanical model of electromagnetic fields and charged particles may precede its realization, which, it should be noted anticipating, is already known.

The following denotations are used below: $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$. The summation over recurrent index is implied throughout.

1. Mechanics of a medium

A medium is characterized by the volume density $\zeta(\mathbf{x},t)$ and the velocity $\mathbf{u}(\mathbf{x},t)$ of the motion of its elements as functions of their position in space $\mathbf{x}$ and time t. Kinematics of the medium is specified by the continuum equation

$$\partial_t\zeta + \partial_i(\zeta u_i) = 0 \qquad (1.1)$$

which puts in differentals the conservation of the medium material in a moving finite volume V:

$$\int_V \zeta d^3x = \text{const.} \qquad (1.2)$$

In the dynamic equation

$$\zeta\partial_t u_i + \zeta u_k\partial_k u_i = \partial_k\sigma_{ik}, \qquad i,k = 1,2,3 \qquad (1.3)$$

the stress tensor σ_{ik} has the meaning of the i-th component of the force, acting on the k-th face of the elementary cube from the adjoining element of the medium. The particular type of the medium or of the motion is specified by the function σ_{ik} of $\zeta(\mathbf{x},t)$ and $\mathbf{u}(\mathbf{x},t)$.

2. Incompressible fluid

The incompressible fluid represents the simplest and at the same time the special case: here the stress tensor does not depend on the strain. In the course of the motion it behaves adjusting itself to the velocity field. In general, we have for the fluid:

$$\sigma_{ik} = -p\delta_{ik} \qquad (2.1)$$

where p is an unknown function of $\mathbf{x}$ and t. Then from (1.3),(2.1) the dynamic equation of the inviscid incompressible fluid is

$$\zeta\partial_t u_i + \zeta u_k\partial_k u_i + \partial_i p = \mathbf{0}. \qquad (2.2)$$

In stationarity, when $\partial_t\mathbf{u} = \mathbf{0}$, this equation can be integrated — along the stream line or along the line of vorticity:

$$\zeta u^2/2 + p = \text{const.} \qquad (2.3)$$

The Bernoulli equation (2.3) demonstrates explicitly what was said before: pressure function $p(\mathbf{x},t)$ adjusts itself to the velocity field $\mathbf{u}(\mathbf{x},t)$.

3. Jelly

We are interested in a medium, whose elements experience only small displacement $\mathbf{s}(\mathbf{x},t)$ from its initial position $\mathbf{x}$. Notice, that here we passed over to the Lagrange representation, whereas the one used before was Eulerian. Thus, $\mathbf{u}(\mathbf{x},t)$ in (1.1), (1.3) belongs to $\mathbf{u}$ at the current, or given point $\mathbf{x}$, but not to the initial position of the medium element. For small displacements the distinction between the two representations becomes inessential. One may assume in this case

$$\mathbf{u} = \partial_t\mathbf{s}. \qquad (3.1)$$

For small velocities we neglect in (1.3) quadratic terms:

$$\zeta\partial_t u_i = \partial_k\sigma_{ik}. \qquad (3.2)$$

For a jelly-like medium, which is incompressible though liable to shear deformations, the stress tensor can be in the following way resolved in terms of the Hooke's law:

$$\sigma_{ik} = \mu(\partial_i s_k + \partial_k s_i) - p\delta_{ik}, \qquad (3.3)$$

$$\partial_i s_i = 0. \qquad (3.4)$$

Substitute (3.3),(3.4) and (3.1) into (3.2):

$$\zeta\frac{\partial^2\mathbf{s}}{\partial t^2} = \mu\nabla^2\mathbf{s} - \nabla p. \qquad (3.5)$$

Remark that one may pass over from the general equation (1.3) to the linearized one (3.2) in a rigor way, not neglecting the quadratic terms. In this event, (3.2) would include the stress tensor in the Lagrange representation. While we have the Eulerian one in (1.3). Strictly speaking, it is for the Lagrange stress tensor that the Hooke's law of the type (3.3) is valid [2].

By virtue of (3.4) the first two terms of (3.5) are solenoidal, while the last one is potential. Hence (3.5) breaks into two independent equations:

$$\zeta\frac{\partial^2\mathbf{s}}{\partial t^2} = \mu\nabla^2\mathbf{s} \qquad (3.6)$$

and

$$\nabla p = 0. \qquad (3.7)$$

The d'Alembert equation (3.6) describes propagation in the jelly-like medium of the transverse wave, whose velocity c is given by

$$c^2 = \mu/\zeta. \qquad (3.8)$$

And (3.7) indicates the invariance of the background pressure:

$$p_0 = \text{const.} \qquad (3.9)$$

4. The luminiferous aether

Rewrite motion equation (3.5) of jelly-like medium in terms of (3.1) and (3.8):

$$\zeta\partial_t\mathbf{u} + \zeta c^2\nabla\times(\nabla\times\mathbf{s}) + \nabla p = 0 \qquad (4.1)$$

where the elastic term was transformed according to the general vector relation

$$\nabla(\nabla\cdot) = \nabla^2 + \nabla\times(\nabla\times) \tag{4.2}$$

and the medium incompressibility condition (3.4). Let us define the vector $\mathbf{A}$, $\mathbf{E}$ and scalar φ fields:

$$\mathbf{A} = \kappa c\mathbf{u}, \tag{4.3}$$
$$\zeta\varphi = \kappa(p - p_0), \tag{4.4}$$
$$\mathbf{E} = \kappa c^2\nabla\times(\nabla\times\mathbf{s}) \tag{4.5}$$

where k is an arbitrary constant. Substitute (4.3)–(4.5) into (4.1):

$$\partial_t\mathbf{A}/c + \mathbf{E} + \nabla\varphi = \mathbf{0}. \tag{4.6}$$

With acoount of (3.1) the medium incompressibility (3.4) gives:

$$\nabla\cdot\mathbf{A} = 0. \tag{4.7}$$

Differentiate (4.5) with respect to t:

$$\partial_t\mathbf{E} - c\nabla\times(\nabla\times\mathbf{A}) = \mathbf{0}. \tag{4.8}$$

At last, take the divergence of (4.5):

$$\nabla\cdot\mathbf{E} = \mathbf{0}. \tag{4.9}$$

Expressions (4.6)–(4.9) coincide with the respective Maxwell's equations with the Coulomb gauge (4.7) in the absence of the electric charge.

5. The external force

We introduce formally into (4.1) the density of an external force:

$$\zeta\partial_t\mathbf{u} + \zeta c^2\nabla\times(\nabla\times\mathbf{s}) + \nabla p - \zeta\mathbf{f} = \mathbf{0}. \tag{5.1}$$

Then we have instead of (4.5)

$$\mathbf{E} = \kappa[c^2\nabla\times(\nabla\times\mathbf{s}) - \mathbf{f}]. \tag{5.2}$$

And equations (4.8),(4.9) will be rewritten as

$$\partial_t\mathbf{E} - c\nabla\times(\nabla\times\mathbf{A}) + \kappa\partial_t\mathbf{f} = 0, \tag{5.3}$$
$$\nabla\cdot\mathbf{E} = -\kappa\nabla\cdot\mathbf{f}. \tag{5.4}$$

With the definitions (4.3),(4.4),(5.2) equation (5.1) reproduce the form (4.6). Redefine the mass force $\mathbf{f}(\mathbf{x},t)$ via the source functions ρ and $\mathbf{j}$:

$$4\pi\mathbf{j} = \kappa\partial_t\mathbf{f}, \tag{5.5}$$
$$4p\rho = -\kappa\nabla\cdot\mathbf{f}. \tag{5.6}$$

From the definitions (5.5),(5.6) we get for (5.3),(5.4):

$$\partial_t\mathbf{E} - c\nabla\times(\nabla\times\mathbf{A}) + 4\pi\mathbf{j} = \mathbf{0}, \tag{5.7}$$
$$\nabla\cdot\mathbf{E} = 4\pi\rho. \tag{5.8}$$

Thus, redefining the vector and scalar fields via (4.3),(4.4),(5.4),(5.5),(5.6) and the elastic coefficient via (3.8) we may obtain from equation (5.1) with an external force the full system of Maxwell's electromagnetic equations (4.6),(5.7),(5.8) with the Coulomb gauge (4.7).

Relations (5.4)–(5.6) redefine the stress source.

6. Charges and particles

Cut out of the elastic medium a ball of the volume V_0 and insert instead a ball of the volume V, extracted from another place of the medium. In case $V < V_0$, the edges should be drawn in and cemented together. Such operation is also possible for the incompressible meduim, provided that it is liable to shear deformations and the pieces of the medium material extracted were interchanged. In the outcome, the medium experiences a radial shift s so that at the distances far enough from the center $\mathbf{x}'$ of the inclusion

$$V - V_0 = 4\pi r^2 s. \tag{6.1}$$

That is, the spherical inclusion produces the potential displacement field

$$4\pi\mathbf{s} = -(V - V_0)\nabla(1/|\mathbf{x}-\mathbf{x}'|). \tag{6.2}$$

Take the divergence of (6.2):

$$\nabla\cdot\mathbf{s} = \Delta V\delta(\mathbf{x}-\mathbf{x}') \tag{6.3}$$

where

$$\Delta V = V - V_0. \tag{6.4}$$

From (6.2) and the deformational part of (3.3) we will find the stresses, produced by the dilatation center (6.3) in the elastic medium:

$$\sigma_{ik} = \mu(\partial_i s_k + \partial_k s_i) = \mu\Delta V\frac{(r^2\sigma_{ik} - 3x_i x_k)}{2\pi r^5}. \tag{6.5}$$

Then, the stress at an area element perpendicular to the radius:

$$n_k = x_k/r, \tag{6.6}$$
$$\sigma_i = \sigma_{ik}n_k = -\mu\Delta V x_i/(\pi r^4). \tag{6.7}$$

The occurence of the dilatation center (6.3) does not affect the Maxwell's equations, since the stresses (6.5) produced by him leads to no body force: $\partial_k\sigma_{ik} = 0$ everythere except $\mathbf{x}'$.

Let, based on the spherical inclusion, the complex stress source be formed in the Maxwell medium, which produces in it the field p of the hydrostatic pressure:

$$\kappa(p - p_0) = q\zeta\phi(|\mathbf{x}-\mathbf{x}'|) \tag{6.8}$$

where q is the strength of the stress source, $\phi(r)$ a function, which tends to zero when r goes to infinity.

The pressure field (6.8) is equalized by the external force according to (5.1):

$$\nabla p = \zeta \mathbf{f}. \tag{6.9}$$

The pressure thrust on the unit area (6.6):

$$(p - p_0)\frac{\delta_{ik} x_k}{r} = (p - p_0)\frac{x_i}{r}. \tag{6.10}$$

At the boundary R of the inclusion (6.10) is balanced by the elastic stress (6.7) of the medium

$$-\sigma_i(R) = [p(R) - p_0] x_i / R \tag{6.11}$$

backing thus the dilatation center (6.3). When (6.11) is valid the ball of the volume V can be taken out of the core of the stress center, leaving there a void (see the figures below).

$$p \longleftarrow \Big) \longrightarrow \sigma \qquad\qquad p \longrightarrow \Big(\longleftarrow \sigma$$

$\Delta V < 0$ The proton

$\Delta V > 0$ The electron

The form (6.8) can be derived from the Maxwell's equation (4.6), which being written in mechanical variables (4.3),(4.4) looks as:

$$\zeta \partial_t \mathbf{u} + \zeta \mathbf{E}/\kappa + \nabla p = 0. \tag{6.12}$$

We consider radial pulsation of the inclusion's core. Such process is possible in the pair of distant hollow inclusions of the opposite sign, formed by the method above described. The disturbance δV of the inclusion's volume $V(t)$ is assumed to be coupled with a variation δq of the stress source's strength $q(t)$, the form $\phi(\mathbf{x} - \mathbf{x}')$ of the pressure field produced by it persisting. The space and time variables can be separated in (6.8) since the longitudinal stress transmits in the incompressible medium instantaneously. Substitute (6.2) and (6.8) in (6.12) taking into account (3.1) and that (6.9) is valid in statics:

$$-\kappa \partial_t^2 \delta V \nabla(1/|\mathbf{x} - \mathbf{x}'|) + 4\pi \delta q \nabla \phi(\mathbf{x} - \mathbf{x}') = 0. \tag{6.13}$$

It can be concluded from (6.13) that

$$\phi = 1/|\mathbf{x} - \mathbf{x}'| \tag{6.14}$$

$$4\pi \delta q = \kappa \partial_t^2 \delta V. \tag{6.15}$$

Thus in the Maxwell medium the stress source associated with the dilatation center produces the pressure field (6.8), whose form (6.14) is determined by the form (6.2) of the radial displacement field.

Sustituting (6.7) and (6.8) with (6.14) into (6.11) and regarding for (3.8) we will find the relation between the strength q of the stress center and the strength ΔV of the dilatation center:

$$\pi R^2 q = \kappa c^2 \Delta V. \tag{6.16}$$

We assume that the charged particle can be viewed as a compound of the mass

$$m = \zeta V \tag{6.17}$$

and the pure charge

$$\pi R_e^2 q_e = k c^2 \Delta V_e \tag{6.18}$$

where the mass of the electron (positron) can be taken

$$m_e = \zeta \Delta V_e. \tag{6.19}$$

At last, substituting (6.14),(6.8) into (6.2),(6.4) and then using in it (6.18), we may write out explicitly the interconnection between the perturbation fields generated in the Maxwell medium by the mechanical analog of the electric charge:

$$4\zeta c^2 \mathbf{s} = -R_e^2 \nabla p = R_e^2 \zeta \mathbf{E}/\kappa. \tag{6.20}$$

7. Plasticity

Relations (5.2),(5.5) redefine the external field. We write down the Maxwell's equation (5.7) thus obtained in the semimechanical form:

$$\partial_t \mathbf{E} - \kappa c^2 \nabla \times (\nabla \times \mathbf{u}) + 4\pi \mathbf{j} = \mathbf{0}. \tag{7.1}$$

Equation (7.1) specifies the dynamic relation of the external force $-\zeta \mathbf{E}/\kappa$ from (6.12) with the solenoidal displacement field and other external force, which are absent in statics. Multiplying (7.1) by $\zeta\delta/\kappa$ and taking into account (3.1) gives:

$$-\zeta \delta \mathbf{E}/\kappa + \zeta c^2 \nabla \times (\nabla \times \delta \mathbf{s}) = 4\pi \zeta \delta t \mathbf{j}/\kappa. \tag{7.2}$$

We see from (7.2) that the displacement grows indefinitely with time owing to the external force, which looks exactly as

$$4\pi \zeta \delta t \mathbf{j}/\kappa. \tag{7.3}$$

Such a behavior is typical for the plastic flow of a medium. Commonly in continuum mechanics the law of plasticity is formulated in terms of the stress tensor and the elastic strain. A more advanced model of the substratum incorporates all the "external" forces as its internal stress sources. In this event the Maxwell's equation (7.1) represents a contraction of a peculiar kind of the Prandtl-Reuss model of the elastic-ideal-plastic medium.

The internal stress sources are provided by the defects of the elastic continuum (see section 6), and the free motion of the defects in the elastic medium gives the mechanism of its plasticity. In those terms the substratum is usually described in physics. The occurence of the defects is indicated by that the right-hand side of the incompatibility relation for the strain field is nonvanishing. In the present context it is (6.3). Apart from the drift, in stochastic environs the defect may split and diffuse forming a defect's gas. Its density ζ is given by the distribution of the singularity (6.3) in the medium:

$$\nabla \cdot \mathbf{s} = \zeta(\mathbf{x}) \tag{7.4}$$

where

$$\int \zeta(\mathbf{x}')d^3x' = \Delta V. \tag{7.5}$$

However, in the Maxwell medium it is the source function $\rho(\mathbf{x}')$ from (5.6) that is used as a defect's gas density. The part of the incompatibility relation is played by (5.8). In this event the Coulomb gauge (4.7) remains to be valid.

Taking the divergence of (7.1) and using (5.8) we get the continuum equation for the defect's gas:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{7.6}$$

where

$$\mathbf{j} = \rho \mathbf{v} \tag{7.7}$$

$\mathbf{v}$ is the velocity of the defect's flow. The differential equation (7.6) expresses the detailed conservation of the strength q of the defect's gas:

$$\int\limits_{V'} \rho d^3x = \mathrm{const} \tag{7.8}$$

where V' is a moving finite volume of the defect's gas. The kinematic equation (7.6) should be supplemented by the respective dynamic equation

$$\rho \partial_t v_i = \partial_k \sigma'_{ik} + \rho f'_i \tag{7.9}$$

where σ'_{ik} is the stress tensor of the defect's gas. Its dependence on r is generally defined in quantum mechanics. Equation (7.9) with the functions $\sigma'_{ik}(\rho)$ and $f'_i(\mathbf{v}, \mathbf{E}, \mathbf{u})$ closes up the set of Maxwell's equations (6.12),(5.8) and (7.1) with (7.7).

Therefore, the substratum material can be carried over only as a result of the motion of the defect's core. Beyond it the flow is concerned with the velocity of the displacement of a medium element from the equilibrium position but not with the occurence at this place of a convective flow.

8. Integrals of motion

Return to the model of the solid elasticity section 4. To obtain the energy integral we will proceed such as it is usually done in classical mechanics. Multiply equation (4.1) by $\mathbf{u}$

$$\zeta \mathbf{u} \cdot \partial_t \mathbf{u} + \zeta c^2 \mathbf{u} \cdot \nabla \times (\nabla \times \mathbf{s}) + \mathbf{u} \cdot \nabla p = 0$$

and integrate it all over the space. The second and third integrals are taken by parts, supposing that integrands are vanishing at infinity. With the account of the incompressibility (3.4),(3.1) that gives:

$$\frac{1}{2}\zeta \int [\mathbf{u}^2 + c^2(\nabla \times \mathbf{s})^2] d^3x = \mathrm{const}. \tag{8.1}$$

In (8.1) the second term corresponds to the energy of the medium twisting and the first one — to its kinetic energy.

Next we derive another type of the integral of motion. Take the curl of (4.1)

$$\partial_t \nabla \times \mathbf{u} - c_g^2 \nabla \times [\nabla \times (\nabla \times \mathbf{s})] = \mathbf{0}$$

multiply it by $\nabla \times \mathbf{u}$ and integrate all over the space, taking the second integral by parts. In the outcome we get a conservation law in the torsion field:

$$\frac{1}{2}\zeta \int \{(\nabla \times \mathbf{u})^2 + c^2[\nabla \times (\nabla \times \mathbf{s})]^2\} d^3x = \mathrm{const}. \tag{8.2}$$

Relation (8.1) describes the energy conservation for the wave of the torsion

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{s}. \tag{8.3}$$

Consider interaction of the two domains $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ of the medium twisting (two torsion fields). From (8.1) their interaction energy is

$$U_{12} = 4\zeta c^2 \int \boldsymbol{\omega}^{(1)} \cdot \boldsymbol{\omega}^{(2)} d^3x. \tag{8.4}$$

Consequently, the wave of like directions $\boldsymbol{\omega}^{(1)} \cdot \boldsymbol{\omega}^{(2)} > 0$ repell each other, the waves of opposite directions $\boldsymbol{\omega}^{(1)} \cdot \boldsymbol{\omega}^{(2)} < 0$ are attracted one to another.

However, the elastic energy is not the only measure of interaction. According to the integral (8.2) of motion, besides the strength, the shape of the torsion field is also of significance. The integral (8.2) of motion in terms of the torsion (8.3) is written as:

$$\frac{1}{2}\zeta_0 \int [(\partial_t \boldsymbol{\omega})^2 + c^2(\nabla \times \boldsymbol{\omega})^2] d^3x = \mathrm{const}. \tag{8.5}$$

The kinetic term of (8.5) is just the enstropy.

9. The work of the external force and the elastic energy

Consider dynamic equation (3.2) with the density $\zeta \mathbf{f}$ of an external force:

$$\zeta \partial_t u_i = \partial_k \sigma_{ik} + \zeta f_i. \tag{9.1}$$

Multiply it by u_i:

$$\frac{1}{2}\zeta \frac{\partial \mathbf{u}^2}{\partial t} - u_i \partial_k \sigma_{ik} = \zeta f_i u_i. \tag{9.2}$$

In (9.2) there are the densities: in the right — of the instantaneous work (power) of the external force, the second term in the left is the elastic energy of the medium, the first term in the left is the kinetic energy of the medium. Proceeding from this the following energy conservation law of the system can be written down:

$$T + U = W \tag{9.3}$$

where W is the work of the external force, U the elastic and T the kinetic energy of the medium. For the equilibrium process in the linear elastic medium W and U appears to be simply related. Consider a model illustration.

Under the action of the weight mg a spring is stretched by a length s. In the equilibrium

$$ks = mg \tag{9.4}$$

where k is the elastic constant of the spring. In this system:

$$W = mgs = ks^2, \tag{9.5}$$

$$U = \int kzdz = ks^2/2. \tag{9.6}$$

In this way we get the relation, which is known in the linear elasticity as Clapeyron theorem:

$$U = W/2. \tag{9.7}$$

Formula (9.7) is readily derived from general equation (9.2), supposing in it a linear dependence of $\sigma_{ik}(\mathbf{z})$ on the current value $\mathbf{z}$ of the displacement and taking the external force in the equilibrium at the final value $\mathbf{s}$ of the displacement:

$$\zeta f_i = -\partial_k \sigma_{ik}. \tag{9.8}$$

Multiplying (9.2) by dt and d^3x, and integrating it over all volume and over t thus, that $\mathbf{z}$ changes from zero to $\mathbf{s}$, we get for (9.3):

$$T = \frac{1}{2}\int \rho \mathbf{u}^2 d^3x, \tag{9.9}$$

$$U = -\frac{1}{2}\int s_i \partial_k \sigma_{ik} d^3x = \frac{1}{2}\int \sigma_{ik}\partial_k s_i d^3x, \tag{9.10}$$

$$W = \int \zeta f_i s_i d^3x = \int \sigma_{ik}\partial_k s_i d^3x. \tag{9.11}$$

Comparing (9.10) with (9.11) confirms (9.7). Supposing that in the quasistatic process the kinetic energy is gradually withdrawn from the circulation, one may substitute (9.7) into (9.3):

$$T - U = 0 \tag{9.12}$$

So, by the Clapeyron theorem (9.12) for the stress sources the signs of the interaction energies in formulas, such as (8.4), obtained from the integrals of motion should be changed to the opposite.

10. Electrostatic interaction

Consider the Maxwell medium as described by equation (4.6), which we take in the mechanical form (6.12). Substituting (2.1) in (9.10) we get the elastic energy of the medium gained owing to the work of the external force $-\zeta\mathbf{E}/\kappa$:

$$U = \frac{1}{2}\int \mathbf{s}\cdot\nabla p d^3x = -\frac{1}{2}\int (p - p_0)\nabla\cdot\mathbf{s} d^3x. \tag{10.1}$$

In the incompressible medium (3.4) $\nabla\cdot\mathbf{s} = 0$. Therefore, the only condition that (10.1) be nonzero is the occurence of the dilatation center (6.3) in the core of the charge. In that case

$$U = -\frac{1}{2}(p - p_0)\Delta V. \tag{10.2}$$

Substitute in (10.2) relations (4.4) and (6.16) taken in the form (6.18):

$$-k^2c^2U = \frac{1}{2}\pi\zeta R_e^2 q\varphi. \tag{10.3}$$

Assuming for the arbitrary dimensional coefficient κ

$$\kappa^2 c^2 = \pi\zeta R_e^2 \tag{10.4}$$

from (10.3) we arrive at the well-known expression for the energy — U of the electrostatic field:

$$-U = \frac{1}{2}q\varphi. \tag{10.5}$$

The sign of (10.5) answers the conditions of the Clapeyron theorem (9.12). Using (10.4) in (6.18) we get with due regard for the additivity of the charge:

$$q = \zeta\Delta V/\kappa. \tag{10.6}$$

Hence, the mechanical meaning of the Coulomb interaction consists in the work against pressure field (6.8),(6.14) needed to create in it the dilatation center (6.3). By (6.20) the external force and the displacement are directed contrariwise. That is why the stress centers of the same sign repell, and the opposite ones attract to each other.

In the conditions of the Clapeyron theorem the work W is computed directly as the product of the force by the displacement $\mathbf{s} = \mathbf{u}\Delta t$ of the substratum element. According to (9.7),(10.5):

$$W = -\zeta\int \mathbf{E}\cdot\mathbf{s} d^3x/\kappa = -q\varphi. \tag{10.7}$$

On the other side, the potential energy (10.7) implies the force

$$-\nabla W = q\nabla\varphi \tag{10.8}$$

which acts upon the defect, or, generalizing it to the defect's gas — the force density

$$\rho\nabla\varphi. \tag{10.9}$$

Then we may consider the work performed on the displacement $\mathbf{v}\Delta t$ of the elements of the defect's gas. The respective instantaneous work (power) is given by

$$\rho\mathbf{v}\cdot\nabla\varphi. \tag{10.10}$$

Compare (10.10) and (10.5) with the terms of the well-known integral of Maxwell's equations

$$(8\pi)^{-1}\partial_t\int[(\nabla\times\mathbf{A})^2+\mathbf{E}^2]d^3x = -\int\mathbf{j}\cdot\mathbf{E}d^3x \tag{10.11}$$

(see Appendix A). When $\mathbf{E}=-\nabla\varphi$ the second term of the left-hand part of (10.11) corresponds to the right-hand part of (10.5), the right-hand part of (10.11) equals to the volume integral of (10.10),(7.7).

In the absence of the charge, (10.11) reduces to the integral of motion of the type (8.2), or (8.5), which is formally not the energy integral of the substratum. However, as we have found, the terms of (10.11) have the meaning of the energy and the work, performed upon the defect's gas. And they are related with the elastic energy of the substratum by virtue of (10.1).

11. The Lorentz force

Let us find the force acted on a charge q_1, which moves in the field φ_2, $\mathbf{A}^{(2)}$. With this end we will define all components necessary to construct the Lagrangian. The cross term singled out from the right-hand side of (10.5) with regard for the Betti's reciprocity theorem provides us with the static interaction potential:

$$q_1\varphi_2=\Delta V_1(p_2-p_0). \tag{11.1}$$

Recall that (10.5) was derived integrating (6.12) over the coordinate $\mathbf{s}$ conjugate to the external force $-\zeta\mathbf{E}/\kappa$, the procedure expounded in section 9 being used. Another component of the Lagragian will be found considering the external force (7.3), which governs the plastic flow. Manipulating (7.1),(6.12) the following integral can be obtained:

$$\frac{1}{2}\int[(\partial_t\mathbf{u})^2+c^2(\nabla\times\mathbf{u})^2d^3x = 4\pi\int\left[\int\mathbf{j}d\mathbf{u}/\kappa\right]d^3x \tag{11.2}$$

(see Appendix B). Multiply (11.2) by $\zeta(\delta t)^2$:

$$\frac{1}{2}\zeta\int[(\partial_t\delta\mathbf{s})^2+c^2(\nabla\times\delta\mathbf{s})^2]d^3x = 4\pi\zeta\delta t\int\left[\int\mathbf{j}d\delta\mathbf{s}/\kappa\right]d^3x. \tag{11.3}$$

The left-hand side of (11.3) reproduces precisely (8.1),(3.1). The right-hand side of (11.3) is the work performed by the force (7.3) on the path

$$\delta\mathbf{s}=\mathbf{u}\delta t. \tag{11.4}$$

Multiply (11.2) by $\kappa^2(4\pi)^{-1}$ taking into account the definition (4.3):

$$(8\pi)^{-1}\int[(\partial_t\mathbf{A}/c)^2+(\nabla\times\mathbf{A})^2]d^3x = \int\left[\int\mathbf{j}\cdot d\mathbf{A}\right]d^3x/c. \tag{11.5}$$

The second term of (11.5) coincides with the magnetic energy from (10.11). Equations (11.3) and (11.5) will be identical provided that we assume

$$\zeta(\delta t)^2=\kappa^2(4\pi)^{-1}. \tag{11.6}$$

The condition (11.6) turns into the scaling relation (10.4) suggesting in it

$$\delta t=\frac{1}{2}R_e/c. \tag{11.7}$$

Therefore the mechanical meaning of the magnetic interaction consists in the work of the external force (7.3) on the field (11.4) of the elastic deformation, both specified for the reference length R_e of the Maxwell medium.

Substitute (11.7) in the expression (7.3),(7.7) and take into account (10.4). Then, for the case of the point charge

$$\rho=q\delta(\mathbf{x}-\mathbf{x}') \tag{11.8}$$

integrating it over all the volume gives

$$2\kappa cq\mathbf{v}/r_e. \tag{11.9}$$

On the other hand, substitute (11.7) into (11.4):

$$\delta\mathbf{s}=\frac{1}{2}\mathbf{u}\frac{r_e}{c}. \tag{11.10}$$

In the conditions of the Clapeyron theorem the work is calculated immediately as the product of the external force (11.9) by the path (11.10):

$$W=\kappa q\mathbf{v}\cdot\mathbf{u}. \tag{11.11}$$

By (9.7),(9.12) the potential energy of the system equals minus half the (11.11). Singling out from the expression thus obtained the cross terms, we will find with due regard for the Betti's reciprocity theorem the interaction potential:

$$-q_1\mathbf{v}^{(1)}\cdot\mathbf{A}^{(2)}/c=-\zeta\Delta V_1\mathbf{v}^{(1)}\cdot\mathbf{u}^{(2)}. \tag{11.12}$$

Expressions (11.1) and (11.12) fully account for the potential energy of the defect. By the rules of composing the Lagrangian, they should be subtracted from the kinetic energy of the defect. Dropping for the sake of simplicity the indices we have in the mechanical form:

$$\frac{1}{2}mv^2+\zeta\Delta V\mathbf{v}\cdot\mathbf{u}-\Delta V(p-p_0) \tag{11.13}$$

where the mass m can be evaluated according to (6.17) or (6.19). The Lagrangian (11.13) provides an external force $q\mathbf{f}'(\mathbf{v}, \mathbf{E}, \mathbf{u})$ for equation (7.9) and thus closes up the set of Maxwell's equations. At last, the power is given by:

$$q\mathbf{f}' \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v} = \partial_t \left(\frac{1}{2} m v^2 \right). \tag{11.14}$$

Make sure that the left-hand side of (11.14) matches minus right-hand side of (10.11) with (11.8). Using (11.14) in (10.11), the latter can be turned into an integral of motion, do not forgetting in this that (11.14) is concerned with the cross terms of (10.11).

12. Concluding remarks

Thus classical electromagnetism fits exactly the structure of linear elasticity. The seat and the bearer of the Maxwell's equations appears to be a solid-like medium, which is tough to compression yet liable to shear deformations. Although only some of the features of electromagnetism are realized in the solid elastic medium, all of them can be described consistently in the language of the theory of linear strains and stresses of this medium. In order to map the feature, which is not realized in the bounds of solid medium, it is sufficient to introduce in the dynamic equation the term of the external force. In spite of its simulative character, such a technique is justified by that the source of the unknown force can be attached to a defect, which is really existent in the solid medium.

Macroscopically the electromagnetic substratum possesses the properties of the elastic-ideal-plastic body. However, in order to decribe the plasticity, we need to introduce no other entities beyond the elastic medium. For, it is sufficient to consider the mechanism of moving about in the medium of the defects, which create its tension.

Certain difficulties arise in connection with the charged particle represented as the superposition of the mass (6.17) and elementary charge (6.18),(6.19). That conception was adopted in order to avoid in (6.16) the dependence of the charge q on the size R of a particle. To substantiate such the approach we will consider the static interaction of the two point charges q_1 and q_2. Supposing additivity of ΔV and p, single out the cross terms from the right-hand side of (10.2), its sign being changed in accord with (9.12):

$$\varepsilon_{12} = \frac{1}{2}[\Delta V_1(p_2 - p_0) + \Delta V_2(p_1 - p_0)] \tag{12.1}$$

where p_2 is taken at the point q_1, and p_1 at the point q_2. Substitute the form (6.8),(6.14) in (12.1):

$$\kappa\varepsilon_{12} = \frac{1}{2}\zeta\frac{\Delta V_1 q_2 + \Delta V_2 q_1}{|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|} \tag{12.2}$$

In order that (12.2) has taken the shape of the Coulomb law, we have to admit in it the equality of the summands. Then (10.6) will be valid. Thus the discreteness of the electric charge proves to be the sufficient condition for the reciprocity of interaction to be fulfilled in the Maxwell medium.

At last, take notice of the important aspect concerned with the derivation of the surface equilibrium condition (6.11) at the inclusion's boundary. Here we supposed that the area stresses, or tractions, are due to the pressure and the elastic strain of the medium, but not caused by the external force. Hence, the force factor Vf introduced by us into the motion equation (5.1) should be rather of the kinetic origin — likewise the inertial forces usually regarded in mechanics.

Most of the assumptions and supplements of the elastic model, including the elasticity itself, are realized in the turbulent model of the luminiferous medium [3–5].

Appendix A

The key points for derivation (10.11) from Maxwell's equations (4.6),(5.7) are given below in mechanical form. The curl of the displacement is expressed in terms of local twisting angle (8.3). Vector $\mathbf{E}$ is expanded into the sum of the solenoidal and potential $\mathbf{E}_g$ components. From (5.7) with the account of (4.3),(3.1) and omitting the current we have:

$$\mathbf{E} = \mathbf{E}_g + 2lc^2 \operatorname{curl} \boldsymbol{\omega}, \tag{A.1}$$

$$\operatorname{curl} \mathbf{E}_g = \mathbf{0}, \tag{A.2}$$

Take the curl of (4.6):

$$2\kappa\partial_t^2\boldsymbol{\omega} + \operatorname{curl} \mathbf{E} = 0. \tag{A.3}$$

Substitute (A.1) into the motion equation (A.3):

$$\partial_t^2\boldsymbol{\omega} + c^2 \operatorname{curl} \operatorname{curl} \boldsymbol{\omega} = 0. \tag{A.4}$$

Multiply (A.4) by $\partial_t\boldsymbol{\omega}$, take the integral over the all volume, the second term of (A.4) integrate by parts, supposing that $\omega \cdot \operatorname{curl} \boldsymbol{\omega}$ is vanishing at infinity:

$$\frac{1}{2}\partial_t \int [(\partial_t\boldsymbol{\omega})^2 + (c \operatorname{curl} \boldsymbol{\omega})^2] d^3x = 0. \tag{A.5}$$

This is the conservation law (8.5) in the field of medium torsion, in other words, in the mechanical model of the magnetic field. The essential point for its derivation is the structure of the vortex ring (8.2) of the solenoidal term of the elastic force in the motion equation (A.4).

Now substitute (A.1) into the motion equation (A.3), keeping $\mathbf{E}_g$:

$$\partial_t^2\boldsymbol{\omega} + c \operatorname{curl} [\mathbf{E}_g/(2\kappa c) + c \operatorname{curl} \boldsymbol{\omega}] = 0. \tag{A.6}$$

Multiply (A.6) by $\partial_t\boldsymbol{\omega}$, take the integral over all the volume, the second term integrate by parts, using in it (A.2):

$$\partial_t \int \{(\partial_t \boldsymbol{\omega})^2 + [\mathbf{E}_g/(2\kappa c) + c\,\mathrm{curl}\,\boldsymbol{\omega}]\} d^3x = 0. \quad \text{(A.7)}$$

This is just (10.11), though discounting the current.

Appendix B

Let us derive (11.2) from Maxwell's equations. Differentiate (6.12) with respect to time:

$$\partial_t \mathbf{u} + \partial_t \mathbf{E}/\kappa + \nabla \partial_t p/\zeta = 0. \quad \text{(B.1)}$$

Multiply (B.1) by $d\mathbf{u} = \partial_t \mathbf{u} dt$:

$$\frac{1}{2} d(\partial_t \mathbf{u})^2 + d\mathbf{u} \cdot \partial_t \mathbf{E}/\kappa + d\mathbf{u} \cdot \nabla \partial_t p/\zeta = 0. \quad \text{(B.2)}$$

Multiply (7.1) by $d\mathbf{u}/\kappa$:

$$d\mathbf{u} \cdot \partial_t \mathbf{E}/\kappa - c^2 d\mathbf{u} \cdot \nabla \times (\nabla \times \mathbf{u}) + 4\pi \mathbf{j} \cdot d\mathbf{u}/\kappa = 0. \quad \text{(B.3)}$$

Subtract (B.3) from (B.2) and integrate it all over the volume, taking by parts the integrals of the third term in (B.2) and the second term in (B.3). By virtue of the Colulomb gauge (4.7) with (4.3) the third term of (B.2) vanishes:

$$\int \left[\frac{1}{2} d(\partial_t \mathbf{u})^2 + c^2 (\nabla \times \mathbf{u}) \cdot (\nabla \times d\mathbf{u}) - 4\pi \mathbf{j} \cdot d\mathbf{u}/\kappa \right] d^3x = 0. \quad \text{(B.4)}$$

Finally, integrating (B.4) over $\mathbf{u}$, we get (11.2).

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