PACS №: 04.20.-q; 04.20.Fy

Ludger Hannibal
Fachbereich Physik
Carl v. Ossietzky Universität Oldenburg
D-26111 Oldenburg
hannibal@uni-oldenburg.de

Dirac Theory in Space-Time without Torsion

Abstract

It is shown that the usual quadratic general-covariant Lagrangian for the Dirac field leads to a symmetric, divergence-free energy-momentum tensor in the standard Riemannian framework of space-time without torsion, provided the tetrad field components are the only quantities related to gravitation that are varied independently.

Dirac's theory of spin 1/2 particles forms the foundation of modern elementary particle physics. In this short article I am concerned with the extension of Dirac's theory to include gravity. It is shown that Dirac's theory fits consistently into the framework of Einstein's theory of general relativity as long as the existence of a tetrad field is assumed.

Tetrode [1] in 1928, Weyl [2], and Fock [3] in 1929 were the first authors [4] who extended Dirac's [5] theory of the electron to include gravitation. Tetrode [1] gave a generalization of the special relativistic Lagrangian density in the form

$$\mathcal{H} = \sqrt{-g}\bar{\psi}\left\{\gamma^a {e_a}^\mu \left(i\hbar\partial_\mu - eA_\mu\right) - mc\right\}\psi. \tag{1}$$

Here ${e_a}^k$ is the orthonormal tetrad field with

$$\begin{aligned} {e_a}^\mu {e_b}^\nu \eta^{ab} &= g^{\mu\nu}, \\ {e^a}_\mu {e^b}_\nu \eta_{ab} &= g_{\mu\nu}, \\ {e_a}^\mu {e^b}_\mu &= {\delta_a}^b, \\ {e^a}_\mu {e_a}^\nu &= {\delta_\mu}^\nu \end{aligned} \tag{2}$$

where a, b, c, d denote the Lorentzian tetrad indices, and $\mu, \nu, \kappa, \lambda$ the general covariant coordinate indices. Indices are raised and lowered by the flat space metric tensor $\eta = diag(1, -1, -1, -1)$ for tetrad indices, and by the metric tensor g for coordinate indices. In our notation the tetrad index is always the first index of e, the coordinate index the second one. In his paper Tetrode [1] focused on two aspects of his ansatz (1). First, he discussed that algebraic constraints on the tetrad field are necessary in order to get consistent equations for ψ and $\bar{\psi}$ from $\mathcal{H}$. This reflects the circumstance that (1) is not general-covariant, ${e_a}^k\partial_k$ is not the general-covariant derivative of a spinor. Secondly, he analysed the canonical energy-momentum tensor arising from $\mathcal{H}$ and noted its asymmetry. Fock [3] derived a general-covariant form of the Dirac equation and discussed the canonical energy-momentum tensor, he noted its asymmetry, too. Subsequent analysis by Costa de Beauregard, Weysenhoff and Raabe, and Papapetrou [6] gave a physical interpretation to the antisymmetric part as spin angular momentum, implying the insufficiency of standard general relativity for fields with spin 1/2 [7]. This is the basis of the Einstein-Cartan-Sciama-Kibble [8] theory of space-time with torsion, an overview is given by Hehl et al. [7].

Weyl [2], as Fock, studied the behaviour of the spin under infinitesimal parallel transport and derived the general-covariant Dirac equation. He took a more intimate look at the general covariant Lagrangian density, which is given in his 1950 paper [9] by

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{i}\left\{\bar{\psi}\gamma^a {e_a}^\mu \partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^a {e_a}^\mu\psi\right\} - \frac{1}{i}\sum {e_a}^\mu {o_\mu}^{bc}\bar{\psi}\gamma^d\gamma^5\psi + 2mc\bar{\psi}\psi \tag{3}$$

where the sum extends over all even permutations of $abcd$ of 0123. Compared to (1) the additional term in (3) makes $\mathcal{L}$ a general-covariant scalar, when the coefficients ${o_\mu}^{bc}$ are properly defined in terms of the tetrad field components. In principle this Lagrangian can also be found in the paper by Fock [3]. Speaking in modern mathematical language, Weyl and Fock constructed a spin-structure with covariant differential [10], which in the notation of [11] reads

$$D\psi = d\psi - \sigma\psi, \quad d\psi = (\partial_\mu\psi)\, dx^\mu \tag{4}$$

where the spin connection σ is given by

$$\sigma = -\frac{1}{4}{\omega^a}_b\gamma_a\gamma^b \tag{5}$$

with the linear connection $\omega^a{}_b$ being a matrix of 1-forms:

$$\omega^a{}_b = \gamma^a{}_{bc}\theta^c, \quad \theta^c = e^c{}_\mu dx^\mu. \tag{6}$$

The coefficients

$$\gamma^a{}_{bc} = e^a{}_i e_c{}^\mu D_\mu e_b{}^i \tag{7}$$

with usual covariant differentiation D_k determine the generalized Ricci rotation coefficients [3,10,11]. In the case of vanishing torsion, which was the only relevant case to Weyl and Fock, the coefficients $e_a{}^p o_p{}^{bc}$ of Weyl's notation are identical with the $\gamma^a{}_{bc}$, which are then given by [11]

$$\begin{aligned}\gamma^a{}_{bc} &= \frac{1}{2}\left(b^a{}_{bc} + b_{bc}{}^a - b_c{}^a{}_b\right),\\ b^a{}_{bc} &= e_b{}^\lambda e_c{}^\mu \partial_\lambda e^a{}_\mu - e_c{}^\lambda e_b{}^\mu \partial_\lambda e^a{}_\mu,\end{aligned} \tag{8}$$

so that (3) takes the manifest general-covariant form

$$\begin{aligned}\mathcal{L}^D &= \sqrt{-g}\left\{\frac{i\hbar}{2}\left(\bar\psi\gamma^a D_a\psi - \overline{D_a\psi}\gamma^a\psi\right) - mc\bar\psi\psi\right\}\\ &= -\hbar\Im\left[\bar\psi\left(\gamma^a D_a - \frac{mc}{i\hbar}\right)\psi\right]\end{aligned} \tag{9}$$

where $\Im$ denotes the imaginary part, and the covariant derivatives $D_a\psi$ of a covariant spinor ψ are defined by

$$D\psi = \theta^a D_a\psi. \tag{10}$$

The general covariant Dirac equation arises from

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}^D}{\delta\bar\psi} = i\hbar\gamma^a D_a\psi - mc\psi. \tag{11}$$

The equivalence of the orthonormal tetrad approach with the general-covariant one can be understood from the result that the existence of a spin structure is equivalent to the existence of a global orthonormal tetrad field [15]. It is interesting to see that neither Fock nor Weyl calculated the energy-momentum tensor resulting from the Lagrangian (9). Fock discussed the canonical energy-momentum tensor thoroughly, but mentioned only afterwards, as a note, that the Dirac equation can be derived by variation of a Lagrangian. Weyl, in his 1950 paper [9], was concerned with the gauge principle, he showed that in general relativity the principle of minimal coupling

$$\partial_k\psi \to (\partial_k + ieA_k)\psi \tag{12}$$

remains valid and is associated with the matter field, arising from gauge transformations. He also looked at the difference between metric and metric-affine theories, where in the latter the coefficients of the affine connection are varied independently of the metric, and showed that these theories are not equivalent, but can be made so by the use of Lagrange multipliers [9]. The idea of using Lagrange multipliers was exploited further by Kichenassamy [11], who took the "tentative view" that the Riemannian structure can be maintained if the vanishing of torsion is enforced by the use of Lagrange multipliers. But behind this idea is still the notion that we do not get a symmetric energy-momentum tensor from the Lagrangian (9), an impression which was seemingly created through the discussion by Tetrode, Weyl, and Fock.

The purpose of this short article is to show explicitly that the Lagrangian (9) leads to a symmetric, divergence-free source tensor for the Einstein field equations when the tetrad field components are considered to be the only quantities related to the gravitational field that are varied independently. Statements regarding the symmetry can be found with Utiyama [12], and Spindel [13], but in both places we do not find a full proof. Therefore it seems appropriate to give a full proof of symmetry and divergence-freeness, which is the basis for the consistency of Einstein-Dirac theory compared to Einstein-Cartan-Dirac theory, thereby showing that the introduction of torsion is not kind of unavoidable necessity [14]. Results on the Dirac equation in both theories have been obtained [16,17].

When the metric is expressed by the tetrad field, the variation with respect to the metric has to be replaced by the variation with respect to the tetrad field. For any Lagrangian $\mathcal{L}(g)$ which depends only on the metric g, we can use (2) to define an e-depending Lagrangian

$$\mathcal{L}'(e) = \mathcal{L}(e\eta e^T). \tag{13}$$

The variations of $\mathcal{L}'$ and $\mathcal{L}$ are then related by

$$\frac{1}{2}\left(\frac{\delta\mathcal{L}}{\delta g^{\kappa\lambda}} + \frac{\delta\mathcal{L}}{\delta g^{\lambda\kappa}}\right) = \frac{1}{2}e^a{}_\kappa\eta_{ab}\frac{\delta\mathcal{L}'}{\delta e_b{}^\lambda}. \tag{14}$$

In this case the r.h.s. of (14) is always symmetric and gives an equivalent way to derive the field equations of Einstein-Maxwell theory with classical particles. But if we define the source tensor for the Einstein field equations for the Dirac Lagrangian (9) by

$$\theta_{\kappa\lambda} = \frac{1}{2\sqrt{-g}}e^a{}_\kappa\eta_{ab}\frac{\delta\mathcal{L}^D}{\delta e_b{}^\lambda}, \tag{15}$$

then the symmetry is not evident.

Since the tensor θ is general-covariant, its symmetry is preserved under coordinate transformations. So it suffices to show that the antisymmetric part of the source tensor (15) vanishes for any point $\mathcal{P}_0$ at which we have a local Lorentz frame (LLF) with [18]

$$\begin{aligned}g_{kl}(\mathcal{P}_0) &= \eta_{kl}, \quad e_a{}^k(\mathcal{P}_0) = \delta_a{}^k,\\ (\partial_i g_{kl})(\mathcal{P}_0) &= \left(\partial_i e_a{}^k\right)(\mathcal{P}_0) = 0,\end{aligned} \tag{16}$$

in order to conclude that θ is symmetric. We use indices i, j, k, l to indicate that values are taken in a local Lorentz frame. The following the calculation is carried out without electromagnetic potential for convenience only, the result can be generalized to the case of a nonvanishing potential A in a straightforward way.

First we introduce new coefficients $\Sigma_a{}^{bcd}$ and write the covariant derivative of a spinor in the more convenient form

$$D_a\psi = \left[e_a{}^\mu\partial_\mu - \Sigma_a{}^{bc}{}_d e_b{}^\lambda\left(\partial_\lambda e_c{}^\mu\right)e^d{}_\mu\right]\psi \qquad (17)$$

with

$$\begin{aligned}\Sigma_a{}^{bcd} &= \frac{1}{4}\left(\delta_a{}^b\Gamma^{cd} - \delta_a{}^c\Gamma^{bd} - \delta_a{}^d\Gamma^{bc}\right),\\ \Gamma^{ab} &= \frac{1}{2}\left(\gamma^a\gamma^b - \gamma^b\gamma^a\right).\end{aligned} \qquad (18)$$

We made use of

$$e_c{}^\mu\partial_\lambda e^d{}_\mu = -\left(\partial_\lambda e_c{}^\mu\right)e^d{}_\mu \qquad (19)$$

to derive (17). With $\theta_{ij}^{(0)} = \theta_{ij}(\mathcal{P}_0)$ we have

$$\begin{aligned}\theta_{ij}^{(0)} = \frac{i\hbar}{4}\big[&\bar\psi\gamma_i\partial_j\psi - \overline{\partial_j\psi}\gamma_i\psi\\ &+\partial_k\left\{\bar\psi\left(\gamma^a\Sigma_a{}^k{}_{ij} + \Sigma_a{}^k{}_{ij}\gamma^a\right)\psi\right\}\big]\\ &+\frac{1}{2}\eta_{ij}\mathcal{L}^D(\mathcal{P}_0).\end{aligned} \qquad (20)$$

In an LLF there is no difference between tetrad and coordinate indices. We point out that the derivatives of the tetrad field components lead to a contribution to the tensor θ_{ij} from the spin connection coefficients even in an LLF, hence θ is not identical with the canonical energy-momentum tensor, which does not have this term in the curly brackets {} in (20). The antisymmetric part of the tensor $\theta_{ij}^{(0)}$ is given by

$$\begin{aligned}\theta_{[ij]}^{(0)} = &\frac{i\hbar}{8}\left[\bar\psi\left(\gamma_i\partial_j - \gamma_j\partial_i\right)\psi - \overline{\left(\gamma_i\partial_j - \gamma_j\partial_i\right)\psi}\psi\right]\\ &+\frac{i\hbar}{4}\partial_k\left\{\bar\psi\left(\gamma^a\Sigma_a{}^k{}_{[ij]} + \Sigma_a{}^k{}_{[ij]}\gamma^a\right)\psi\right\}.\end{aligned} \qquad (21)$$

At $\mathcal{P}_0$ the Dirac equation has its special-relativistic form

$$\left(i\hbar\gamma^k\partial_k - mc\right)\psi(\mathcal{P}_0) = 0, \qquad (22)$$

which after multiplication with γ_i can be written as [19]

$$\partial_i\psi(\mathcal{P}_0) = \left(-\Gamma_i{}^k\partial_k - \frac{imc}{\hbar}\gamma_i\right)\psi(\mathcal{P}_0). \qquad (23)$$

This is used to get

$$\begin{aligned}\left(\gamma_i\partial_j - \gamma_j\partial_i\right)\psi &= \frac{1}{2}\Big\{\gamma_i\partial_j - \gamma_j\partial_i\\ &-\gamma_i\left(\Gamma_j{}^k\partial_k + \frac{imc}{\hbar}\gamma_j\right) + \gamma_j\left(\Gamma_i{}^k\partial_k + \frac{imc}{\hbar}\gamma_i\right)\Big\}\psi\\ &= \frac{1}{2}\left(\Gamma_{ji}\gamma^k\partial_k + \gamma^k\Gamma_{ji}\partial_k + 2\frac{imc}{\hbar}\Gamma_{ji}\right)\end{aligned} \qquad (24)$$

which leads to

$$\begin{aligned}\theta_{[ij]}^{(0)} = &\frac{i\hbar}{16}\Big[\bar\psi\left(\Gamma_{ji}\gamma^k\partial_k + \gamma^k\Gamma_{ji}\partial_k + 2\frac{imc}{\hbar}\Gamma_{ji}\right)\psi\\ &-\overline{\left(\Gamma_{ji}\gamma^k\partial_k + \gamma^k\Gamma_{ji}\partial_k + 2\frac{imc}{\hbar}\Gamma_{ji}\right)\psi}\psi\Big]\\ &+\frac{i\hbar}{4}\partial_k\left\{\bar\psi\left(\gamma^a\Sigma_a{}^k{}_{[ij]} + \Sigma_a{}^k{}_{[ij]}\gamma^a\right)\psi\right\}\\ &= \frac{i\hbar}{16}\Big[\bar\psi\left(\Gamma_{ji}\gamma^k + \gamma^k\Gamma_{ji}\right)\partial_k\psi +\\ &\overline{\partial_k\psi}\left(\Gamma_{ji}\gamma^k + \gamma^k\Gamma_{ji}\right)\psi\Big]\\ &+\frac{i\hbar}{16}\partial_k\left\{\bar\psi\left(\gamma^k\Gamma_{ij} + \Gamma_{ij}\gamma^k\right)\psi\right\} = 0,\end{aligned} \qquad (25)$$

since

$$\Sigma_a{}^b{}_{[cd]} = \frac{1}{4}\delta_a{}^b\Gamma_{cd} \text{ and } \Gamma_{ij} = -\Gamma_{ji} \qquad (26)$$

and because the terms containing the mass cancel. Hence θ is symmetric. This symmetry also holds when an electromagnetic potential is included. Tetrode [1] already noticed that the antisymmetic part of the full (canonical) energy-momentum tensor does not depend on the potential. This can be seen also from (25). If we replace $\partial_\mu \to \partial_\mu + ieA_\mu$ in the rectangular brackets [] in (25), the additional terms cancel.

Next we show that θ is also divergence-free if the Dirac equation is satisfied. Since we do not know anything about the second order derivatives of the tetrad field components in an LLF, we have to show that their contributions cancel. We start with the full energy-momentum tensor. Since from (9) $\mathcal{L}^D \equiv 0$ for any solution of the Dirac equation (11) we have

$$\theta^\kappa{}_\lambda = \frac{1}{2\sqrt{-g}}e_b{}^\kappa\frac{\delta\mathcal{L}^D}{\delta e_b{}^\lambda} = -\frac{\hbar}{2}\Im\left(\tilde\theta^\kappa{}_\lambda\right) \qquad (27)$$

with

$$\begin{aligned}\tilde\theta^\kappa{}_\lambda = &\bar\psi\gamma^a e_a{}^\kappa\partial_\lambda\psi - \left(\bar\psi\gamma^a\Sigma_a{}^{bc}{}_d\psi\right)\\ &\times\left\{e_b{}^\kappa\left(\partial_\lambda e_c{}^\mu\right)e^d{}_\mu - e_b{}^\nu\left(\partial_\nu e_c{}^\kappa\right)e^d{}_\lambda\right\}\\ &+\frac{1}{\sqrt{-g}}e_c{}^\kappa\partial_\nu\left(\sqrt{-g}\bar\psi\gamma^a\Sigma_a{}^{bc}{}_d\psi e_b{}^\nu e^d{}_\lambda\right)\\ &= \bar\psi\gamma^a e_a{}^\kappa\partial_\lambda\psi - \bar\psi\gamma^a\Sigma_a{}^{bc}{}_d\psi\\ &\times\left\{e_b{}^\kappa\left(\partial_\lambda e_c{}^\mu\right)e^d{}_\mu - e_b{}^\nu\left(\partial_\nu e_c{}^\kappa\right)e^d{}_\lambda\right\}\\ &+\partial_\nu\left(\sqrt{-g}\bar\psi\gamma^a\Sigma_a{}^{bc}{}_d\psi e_c{}^\kappa e_b{}^\nu e^d{}_\lambda\right)\\ &-\frac{1}{\sqrt{-g}}\left(\partial_\nu\sqrt{-g}e_c{}^\kappa\right)\left(\bar\psi\gamma^a\Sigma_a{}^{bc}{}_d\psi e_b{}^\nu e^d{}_\lambda\right)\\ &= \bar\psi\gamma^a e_a{}^\kappa\partial_\lambda\psi - \left(\bar\psi\gamma^a\Sigma_a{}^{bc}{}_d\psi\right)\\ &\times\left\{e_b{}^\kappa\left(\partial_\lambda e_c{}^\mu\right)e^d{}_\mu - e_c{}^\kappa e_b{}^\nu e^d{}_\lambda\left(\partial_\nu e_f{}^\mu\right)e^f{}_\mu\right\}\\ &+\partial_\nu\left(\sqrt{-g}\bar\psi\gamma^a\Sigma_a{}^{bc}{}_d\psi e_c{}^\kappa e_b{}^\nu e^d{}_\lambda\right)\end{aligned} \qquad (28)$$

where we used the relation

$$\partial_\nu\sqrt{-g} = -\sqrt{-g}\left(\partial_\nu e_f{}^\mu\right)e^f{}_\mu. \qquad (29)$$

We need only calculate $\partial_i \theta^i{}_j$ in order to obtain the covariant divergence in an LLF since $\left(\theta^i{}_{j;i}\right)(\mathcal{P}_0) = \left(\partial_i \theta^i{}_j\right)(\mathcal{P}_0)$. We use (11) and (17) to obtain

$$\begin{aligned} \partial_i \left(\bar\psi \gamma^a e_a{}^i \partial_j \psi\right) &= \overline{\left(\gamma^a e_a{}^i \partial_i \psi\right)} \partial_j \psi + \bar\psi \left(\partial_i e_a{}^i\right) \gamma^a \partial_j \psi \\ &+ \bar\psi \partial_j \left(\gamma^a e_a{}^i \partial_i \psi\right) - \bar\psi \left(\partial_j e_a{}^i\right) \gamma^a \partial_i \psi \doteq \overline{\left(\frac{mc}{i\hbar}\psi\right)} \partial_j \psi \\ &+ \bar\psi \partial_j \left(\left[\frac{mc}{i\hbar} + \gamma^a \Sigma_a{}^{bc}{}_d e_b{}^l \left(\partial_l e_c{}^k\right) e^d{}_k\right] \psi\right) \\ &\doteq \left(\bar\psi \gamma^a \Sigma_a{}^{bc}{}_d \psi\right) \partial_j \left(e_b{}^l \left(\partial_l e_c{}^k\right) e^d{}_k\right) \end{aligned} \quad (30)$$

where the dot with $\doteq$ indicates that we have omitted terms which vanish in an LLF. Since the coefficients Σ are antisymmetric in the two middle indices,

$$\Sigma_a{}^{bc}{}_d = -\Sigma_a{}^{cb}{}_d \quad (31)$$

the total divergence in $\theta^i{}_j$, the last term in the rectangular bracket in the last line of (27), gives no contribution to the divergence, so with help of (30) we are left with

$$\begin{aligned} \partial_i \tilde\theta^i{}_j &\doteq \left(\bar\psi \gamma^a \Sigma_a{}^{bc}{}_d \psi\right) \left\{\partial_j \left(e_b{}^l \left(\partial_l e_c{}^k\right) e^d{}_k\right)\right. \\ &\left. -\partial_i \left(e_b{}^i \left(\partial_j e_c{}^k\right) e^d{}_k - e_c{}^i e_b{}^l e^d{}_j \left(\partial_l e_f{}^k\right) e^f{}_k\right)\right\} \\ &\doteq \left(\bar\psi \gamma^a \Sigma_a{}^{bc}{}_d \psi\right) \left\{e_b{}^l \left(\partial_j \partial_l e_c{}^k\right) e^d{}_k - e_b{}^i \left(\partial_i \partial_j e_c{}^k\right) e^d{}_k\right. \\ &\left. + e_c{}^i e_b{}^l e^d{}_j \left(\partial_i \partial_l e_f{}^k\right) e^f{}_k\right\} = 0 \end{aligned} \quad (32)$$

where we again left out terms containing only first derivatives of the tetrad field, and used (31). So we have shown

$$\left(\theta^i{}_{j;i}\right)(\mathcal{P}_0) = 0, \quad (33)$$

hence θ is divergence-free. Thus θ has the same properties as for a classical point particle. Although symmetric, θ_{ij} is not identical with the symmetric part of the canonical energy-momentum tensor since that part of Σ, which is symmetric in the last two indices, does not vanish. Thus there is a contribution to the energy-momentum tensor from the spin angular momentum. The properties of θ show that it reduces in an LLF to the Belinfante-Rosenfeld [20] symmetrized energy-momentum tensor of special relativity, which differs from the asymmetric canonical one by a divergence-free spin-part [20, 21]. This is completely in correspondence to the case of the electromagnetic field in the Einstein-Maxwell theory, where the contribution

$$\begin{aligned} \theta^{e.m.}_{\mu\nu} &= \left(\delta \mathcal{L}^{e.m.}/\delta g^{\mu\nu}\right)/\sqrt{-g}, \\ \mathcal{L}^{e.m.} &= -\frac{\sqrt{-g}}{4\mu_0 c} F^{\mu\nu} F_{\mu\nu} \end{aligned} \quad (34)$$

of the electromagnetic field to the energy-momentum tensor is also symmetric with vanishing covariant divergence (in the absence of matter) and reduces in an LLF to the correct gauge-invariant physical energy-momentum tensor, which is identical with the Belinfante-Rosenfeld construction [21]. In the presence of a spin-1/2 field and the electromagnetic field only the sum $\theta_{\mu\nu} + \theta^{e.m.}_{\mu\nu}$ is divergence-free. This is part of a more general result showing that in Einstein-Maxwell-Dirac theory the energy momentum tensors are identical with the Belinfante-Rosenfeld tensors [22].

I have shown that the Lagrangian given by Weyl and Fock for the Dirac field, (9), together with minimal coupling for the electromagnetic potential, leads to a viable formulation of Einstein-Maxwell-Dirac theory in the standard framework of a Riemannian space-time. It is general-covariant, it reduces to the special relativistic Lagrangian in the case of vanishing gravitational field. It leads to a symmetric, divergence-free energy-momentum tensor, which allows for a consistent classical limit. In a local Lorentz frame the energy-momentum tensor reduces to the Belinfante-Rosenfeld symmetric one, in correspondence with the case of the electromagnetic field. As Weyl [9] showed, this theory is not equivalent to a metric-affine theory. The metric-affine theory is equivalent to this theory with the well-known Heisenberg-Pauli-type terms added, which are quadratic in the spinor field [11, 23].

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