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Galilean Covariant Dirac Equation

Contents

Abstract

We present a construction of Galilean covariant Dirac equations. Following the method which led P.A.M. Dirac to the discovery of his famous relativistic wave equation we construct analogues of the Dirac equation using the extended Galilei group as a symmetry group of the spacetime. These equations are not simply the $c \to \infty$ limits of the equation describing relativistic spin 1/2 particle. They exhibit numerous properties shared with the relativistic description.

1. Introduction

The relativistic wave equation, known now as Dirac equation, is one of the greatest achievements of theoretical physics ever made. P.A.M. Dirac found it in 1928 [1] and at that time his aim was to unify special theory of relativity with just born quantum mechanics. Dirac equation, so different from all equations previously known, immediately faced an enormous success. It allowed to understand the fine structure of the hydrogen atom spectrum and explained the nature of spin introduced to quantum physics by W. Pauli in order to describe the atomic periodic system. But the greatest success of the equation, and Dirac himself, consisted in the prediction of the existence of antiparticles. This at first pure speculation, formulated as the "hole theory", tried to justify the presence of negative energy solutions. It became however a physical reality when K.D. Anderson discovered the positron in 1932. This discovery gave the beginning of one of the most powerful physical theory whichever formulated: the quantum electrodynamics. As it is well-known this theory provides us with the most exact theoretical predictions which may be compared with experimental data [2].

In the following we are going to show that Dirac's arguments [3], which led him to his famous equation, have more general meaning. If applied to nonrelativistic, *i.e.*, Galilean invariant, description of quantum phenomena they allow, within a geometric approach, to write down Galilean covariant wave equations which interpretation also requires the existence of both nonclassical degrees of freedom (internal angular momentum) and antiparticles (a concept commonly associated with the relativistic invariance). Here we would like to emphasize that our approach, elaborated in the series of papers [4–7], differs from studies of nonrelativistic first order wave equations which have been done by the others. These investigations, begun by J.-M. Levy-Leblond [8–11] (see also [12] for a recent review), have been developed both in the context of the Dirac equation [13] as well as in the context of the first order wave equation for any spin — the analogues of the Duffin-Kemmer-Petiau [14] and Bhabha equations [15] known from the relativistic field theory. Nevertheless none of these results, although they all are based on geometrical approach and they apply the methods of the extended Galilean symmetry, has taken into account that a comprehensive and careful study of the problem leads to equations which general solutions require the presence of negative energy parts — the property which, as we know now, should be interpreted as a way of description of antiparticles. We are strongly convinced this paper will fill this gap.

2. Galilean Symmetry of Quantum Physics

Symmetry principles play a very important role in quantum physics and it is an unjustified prejudice to think that the spacetime symmetries are important only in the relativistic description while in the case of nonrelativistic symmetries their study leads to trivial conclusions. As a result it is almost forgotten that under Galilean transformations the Schroedinger wave function is not a scalar but it acquires an additional local phase factor. Since the classical papers by V. Bargmann [16] and E. Inönu and E. P. Wigner [17] it is known that wave functions of physical particles cannot belong to the carrier space of vector representations of the Galilei group. Therefore in the construction of nonrelativistic field theory one can not straightforwardly follow the group-theoretical methods elaborated in the framework of the relativistic field theory. The way to get out of the complications is either to apply the formalism of projective representations of the Galilei group (being more complicated than the vector representations) or to postulate that the symmetry of the nonrelativistic spacetime is described by the extended Galilei group [11] and to use its vector representations. Within the latter approach the Newtonian spacetime $M_4 = (t, \mathbf{x})$ is assumed to be embedded in a five dimensional manifold $M_5 = (t, \theta, \mathbf{x})$. Its coordinates — components of a contravariant five vector — transform according to

$$\begin{aligned} t' &= t + b, \\ \theta' &= \theta + \mathbf{v}\cdot\mathcal{R}\mathbf{x} + \frac{1}{2}\mathbf{v}^2 t + \omega, \\ \mathbf{x}' &= \mathcal{R}\mathbf{x} + \mathbf{v}t + \mathbf{a} \end{aligned} \tag{1}$$

where $\mathbf{v}$ denotes the relative velocity of inertial frames, $\mathcal{R}$ is the matrix of three dimensional rotations and b, ω and $\mathbf{a}$ parametrize five dimensional translations. Transformation rules for derivatives — components of a covariant vector — go as follows

$$\begin{aligned} \frac{\partial}{\partial t}' &= \frac{\partial}{\partial t} - \mathbf{v}\cdot\mathcal{R}\nabla + \frac{1}{2}\mathbf{v}^2\frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial\theta}' &= \frac{\partial}{\partial\theta}, \\ \nabla' &= \mathcal{R}\nabla - \mathbf{v}\frac{\partial}{\partial\theta}. \end{aligned} \tag{2}$$

The transformation rules (1) leave invariant the bilinear form

$$\Delta s \;=\; 2\Delta\theta\Delta t - (\Delta\mathbf{x})^2 + \alpha^2(\Delta t)^2 \tag{3}$$

for arbitrary value of α^2, a dimensional parameter which bears the dimension of a velocity squared. It enables us to identify

$$g_{ab} \;=\; \begin{pmatrix} \alpha^2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \qquad a, b = t, \theta, 1, 2, 3 \tag{4}$$

as a covariant metric tensor in the extended spacetime $t, \theta, \mathbf{x}$. Its inverse, *i.e.*, a contravariant metric tensor is

$$g^{ab} \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -\alpha^2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \qquad a, b = t, \theta, 1, 2, 3. \tag{5}$$

It is obvious that having metric tensors we may easily develop the Galilean tensor calculus [18] and to construct invariant expressions both from contravariant as well as from covariant vectors. In particular from (5) and the components of the five-gradient (2) one immediately gets second rank invariant differential operators giving equations which analysis will be the subject of the next section.

Here we would like to mention that the formulation of the Galilean symmetry as a symmetry of the five dimensional spacetime allows to get a very interesting generalization. By rewriting the bilinear form (3) as

$$\Delta s = \left(\alpha\Delta t + \frac{\Delta\theta}{\alpha}\right)^2 - \frac{(\Delta\theta)^2}{\alpha^2} - (\Delta\mathbf{x})^2 \tag{6}$$

we can see that the maximal symmetry group which leaves this form invariant is the pseudoorthogonal de Sitter group $so(1, 4)$ which subgroups are not only the Galilei group and its one parameter central extension but also the Lorentz group. Because of that one should expect that the wave equations which one is going to construct will be, in general, covariant with respect to much larger groups than just the Galilean one. The problem really occurs and it was investigated in various aspects. The investigations known to us are associated both with a construction of field theoretical models in the flat Galilean spacetime [5–7], [13–15], as well as with the "Newtonian"version of general relativity, *i.e.*, with low energy limit of the gravity theory formulated in the curved Galilean spacetime [19, 20].

3. Galilean Covariant Wave Equations

3.1. Analogue of the Klein-Gordon Equation

The formalism developed in the previous section enables us to write down the second rank linear differential operator

$$L_2(\alpha) = \alpha^{-2}\left(2\frac{\partial}{\partial t}\frac{\partial}{\partial\theta} - \triangle\right) - \frac{\partial^2}{\partial\theta^2}, \tag{7}$$

which remains invariant with respect to the transformation rules (2). Having this we can write [7], for a free nonrelativistic scalar particle of mass m and rest energy $\Omega = m\alpha^2$, a following wave equation

$$L_2(\alpha)\psi(t,\theta,\mathbf{x}) = \left(\alpha^{-2}\left(2\frac{\partial}{\partial t}\frac{\partial}{\partial\theta} - \triangle\right) - \frac{\partial^2}{\partial\theta^2}\right) \times \psi(t,\theta,\mathbf{x}) = -\frac{m^2}{\hbar^2}\psi(t,\theta,\mathbf{x}), \tag{8}$$

which, however, does not guarantee neither the nonrelativistic dispersion relation

$$E = \frac{\mathbf{p}^2}{2m} + \Omega, \tag{9}$$

nor a conserved probability current, *i.e.*, a conserved five vector with a positive definite time component.

The invariant operator $L_2(\alpha)$ is nothing else but one of the Casimir operators for the Lie algebra of generators of the extended Galilei group [11]. Therefore the equation (8) expresses the fact that in any irreducible representation of this algebra the Casimir operator is proportional to the unite operator. In order to get a truly irreducible representation we must utilize also the second Casimir operator

$$L_2' = \frac{\partial^2}{\partial\theta^2} \tag{10}$$

which clearly is an invariant second rank differential operator. Assuming, as usually, that wave functions span irreducible representations of the spacetime symmetry group we arrive at a conclusion that besides equation (8) the free nonrelativistic scalar particle of mass m and rest energy $\Omega = m\alpha^2$ satisfies also the equation

$$L_2'\psi(t,\theta,\mathbf{x}) = \frac{\partial^2}{\partial\theta^2}\psi(t,\theta,\mathbf{x}) = -\frac{m^2}{\hbar^2}\psi(t,\theta,\mathbf{x}). \tag{11}$$

Under this condition the general solution of (8) is

$$\psi(t,\theta,\mathbf{x}) = \exp\left(-i\frac{m\theta}{h}\right)\psi_+(t,\mathbf{x}) + \exp\left(i\frac{m\theta}{h}\right)\psi_-(t,\mathbf{x}) \tag{12}$$

where $\psi_+(t,\mathbf{x})$ and $\psi_-(t,\mathbf{x})$ are solutions of the free Schroedinger equation, but with positive and negative masses and rest energies, respectively. It gives for them two dispersion relations

$$E_+ = \frac{\mathbf{p}^2}{2m} + \Omega, \qquad E_- = -\frac{\mathbf{p}^2}{2m} - \Omega \tag{13}$$

and suggests that, analogously to the relativistic field theory, we deal with equations which describe simultaneously particles and antiparticles. This shows that the existence of solutions with negative energy is not a peculiarity of the Poincare invariant equations. These two classes of solutions appear as well in nonrelativistic descriptions.

The solution (12) does not solve however the problem of the existence of a conserved probality current with positively definite probability density which is necessary for a single particle interpretation of the theory. In order to achieve this aim we will follow the original Dirac's scheme.

3.2. Galilean Covariant Dirac Equation

The crucial idea in Dirac's construction was the observation that the existence of a conserved probability current is connected with the fact that in the Schroedinger equation only the first order time derivative is present. This, joined with the requirement of relativistic covariance, led Dirac to the conclusion that the correct relativistic wave equation should be of the first order in all coordinates and should be consistent with the relativistic dispersion relation.

As we are going to follow this way we shall replace (8) and (11) by their first order analogues. Taking the "square root"of (7) we get two first order differential operators

$$L_{1\pm}(\alpha) = \alpha^{-2}\left(\gamma^0 \pm \gamma_5\right)\frac{\partial}{\partial t} + \alpha^{-1}\gamma\cdot\nabla \mp \gamma_5\frac{\partial}{\partial\theta} \tag{14}$$

where γ^0 and γ are the usual Dirac matrices satisfying anticommutation relations

$$\left\{\gamma^i, \gamma^j\right\} = 2g^{ij}, \quad g = diag(1, -1, -1, -1) \tag{15}$$

and $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$. The operators $L_{1\pm}$ are unitarily inequivalent which is a consequence of the fact that the Clifford algebra generated by the metric tensor (5) has two unitarily inequivalent four dimensional representations, each of which may be built from the Dirac matrices. Because of that we get

$$\begin{aligned} &L_{1\varepsilon}(\alpha)\psi_{(\varepsilon\eta)}(t,\theta,\mathbf{x}) \\ &= \left(\alpha^{-2}\left(\gamma^0 + \varepsilon\gamma_5\right)\frac{\partial}{\partial t} + \alpha^{-1}\gamma\cdot\nabla - \varepsilon\gamma_5\frac{\partial}{\partial\theta}\right) \\ &\qquad \times \psi_{(\varepsilon\eta)}(t,\theta,\mathbf{x}) = i\eta\frac{m}{\hbar}\psi_{(\varepsilon\eta)}(t,\theta,\mathbf{x}) \end{aligned} \tag{16}$$

with $\varepsilon, \eta = \pm 1$, as possible candidates to be an equation looked for. This should be completed with the first order equations got from (11) as its "square root"

$$\frac{\partial}{\partial\theta}\,\psi_{(\cdot\ \xi)}(t,\theta,\mathbf{x}) = -i\xi\frac{m}{\hbar}\psi_{(\cdot\ \xi)}(t,\theta,\mathbf{x}) \tag{17}$$

where again $\xi = \pm 1$. It restricts solutions of (16) to

$$\psi_{(\cdot\ \xi)}(t,\theta,\mathbf{x}) = \exp\left(-i\xi\frac{m\theta}{\hbar}\right)\psi_{(\cdot\ \xi)}(t,\mathbf{x}) \tag{18}$$

which, if put into (16), replaces it by the following set of equations

$$\begin{aligned} &\left[\alpha^{-2}\left(\gamma^0 + \varepsilon\gamma_5\right)\frac{\partial}{\partial t} + \alpha^{-1}\gamma\cdot\nabla - i\frac{m}{\hbar}\left(\eta - \varepsilon\xi\gamma_5\right)\right] \\ &\qquad \times \psi_{(\varepsilon\eta\xi)}(t,\mathbf{x}) = 0. \end{aligned} \tag{19}$$

The eight equations above are not independent. Exploiting the properties of Dirac matrices one can show that their solutions, *i.e.* the wave functions $\psi_{(\varepsilon\eta\xi)}(t,\theta,\mathbf{x})$ with $\varepsilon, \eta, \xi = \pm 1$ satisfy the following identities

$$\begin{aligned} \psi_{(+++)} &\sim \gamma_5\psi_{(--+)} \sim \gamma^1\gamma^3\psi^*_{(+--)} \\ &\sim \gamma^0\gamma^2\psi^*_{(-+-)}, \\ \psi_{(---)} &\sim \gamma_5\psi_{(++-)} \sim \gamma^1\gamma^3\psi^*_{(-++)} \\ &\sim \gamma^0\gamma^2\psi^*_{(+-+)}. \end{aligned} \tag{20}$$

Because of that we are left with two independent equations only - here chosen to be

$$\begin{aligned} &\left[\alpha^{-2}\left(\gamma^0 + \gamma_5\right)\frac{\partial}{\partial t} + \alpha^{-1}\gamma\cdot\nabla - i\frac{m}{\hbar}\left(1 - \gamma_5\right)\right] \\ &\qquad \times\psi_{(+)}(t,\mathbf{x}) = 0, \\ &\left[\alpha^{-2}\left(\gamma^0 - \gamma_5\right)\frac{\partial}{\partial t} + \alpha^{-1}\gamma\cdot\nabla + i\frac{m}{\hbar}\left(1 + \gamma_5\right)\right] \\ &\qquad \times\psi_{(-)}(t,\mathbf{x}) = 0 \end{aligned} \tag{21}$$

where $\psi_{(+)}(t,\mathbf{x}) \equiv \psi_{(+++)}(t,\mathbf{x})$ and $\psi_{(-)}(t,\mathbf{x}) \equiv \psi_{(---)}(t,\mathbf{x})$. In the following we shall see that both of them are necessary in order to preserve the invariance of the whole system with respect to discrete symmetries like space reflection P, time reversal T and charge conjugation C.

Using the representation of the Dirac matrices

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma_5 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \gamma^j &= \begin{pmatrix} i\sigma^j & 0 \\ 0 & -i\sigma^j \end{pmatrix} \end{aligned} \tag{22}$$

written in terms of two-dimensional unite and Pauli matrices, we arrive at

$$\begin{aligned} &\left[\alpha^{-2}\begin{pmatrix} 0 & 0 \\ 2\partial_t & 0 \end{pmatrix} + \alpha^{-1}\begin{pmatrix} i\sigma^j\partial_j & \\ 0 & -i\sigma^j\partial_j \end{pmatrix}\right. \\ &\qquad \left. -i\frac{m}{\hbar}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right]\begin{pmatrix} \phi^1_+(t,\mathbf{x}) \\ \phi^2_+(t,\mathbf{x}) \end{pmatrix} = 0 \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\left[\alpha^{-2}\begin{pmatrix} 0 & 2\partial_t \\ 0 & 0 \end{pmatrix} + \alpha^{-1}\begin{pmatrix} i\sigma^j\partial_j & \\ 0 & -i\sigma^j\partial_j \end{pmatrix}\right. \\ &\qquad \left. +i\frac{m}{\hbar}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right]\begin{pmatrix} \phi^1_-(t,\mathbf{x}) \\ \phi^2_-(t,\mathbf{x}) \end{pmatrix} = 0 \end{aligned} \tag{24}$$

as equations for two-dimensional components of $\psi_{(+)}(t,\mathbf{x}) = \left(\phi^1_+(t,\mathbf{x}), \phi^2_+(t,\mathbf{x})\right)$ and $\psi_{(-)}(t,\mathbf{x}) = \left(\phi^1_-(t,\mathbf{x}), \phi^2_-(t,\mathbf{x})\right)$. Only $\phi^1_+(t,\mathbf{x})$ and $\phi^2_-(t,\mathbf{x})$ have a dynamical meaning which means that our system of equations describes 4 degrees of freedom. The solvability conditions of (23) and (24) are the Schroedinger equations containing the rest energy terms and they read

$$\left(i\hbar\partial_t - \frac{\hat{p}^2}{2m} - m\alpha^2\right)\psi_{(+)}(t,\mathbf{x}) = 0 \tag{25}$$

and

$$\left(i\hbar\partial_t + \frac{\hat{p}^2}{2m} + m\alpha^2\right)\psi_{(-)}(t,\mathbf{x}) = 0, \tag{26}$$

respectively. For the plane wave solutions we have $E = p^2/2m + m\alpha^2$ in the case of (25) and $E = -p^2/2m - m\alpha^2$ in the case of (26) which imply that (23) describes a particle of mass m and rest energy $m\alpha^2$, while (24) describes an antiparticle of mass $-m$ and rest energy $-m\alpha^2$, both allowed to be in two spin states. The conserved five-vector current is a linear combination of separately conserved particle and antiparticle currents

$$\begin{aligned} j^t &= A\bar{\psi}_{(+)}\left(\gamma^0 + \gamma_5\right)\psi_{(+)} \\ &\qquad + B\bar{\psi}_{(-)}\left(\gamma^0 - \gamma_5\right)\psi_{(-)}, \\ j^\theta &= -\alpha^2\left(A\bar{\psi}_{(+)}\gamma_5\psi_{(+)} - B\bar{\psi}_{(-)}\gamma_5\psi_{(-)}\right), \\ j^k &= -\alpha\left(A\bar{\psi}_{(+)}\gamma^k\psi_{(+)} - B\bar{\psi}_{(-)}\gamma^k\psi_{(-)}\right) \end{aligned} \tag{27}$$

with the usual notation $\bar{\psi}_{(\cdot)} = \psi^{\dagger}_{(\cdot)}\gamma^0$. Its t-component is explicitly positively definite and, in contradistinction to the relativistic case, none of its components contain the interference terms between positive and negative energy solutions. This means that both sectors described by the model remain separated and that the nonrelativistic particles can not annihilate. Obviously it agrees with the nonrelativistic principle of the particle number, or mass, conservation.

3.3. Nonrelativistic Electromagnetic Fields

In order to confirm the hypothesis that the two equations constructed in the previous section really describe particles and antiparticles one should study their properties in the presence of an electromagnetic field. The model of electromagnetic field in extended Galilean spacetime is known from more than twenty years [4]. It introduces a five-vector of the electromagnetic potential $\mathcal{A}(t,\theta,\mathbf{x}) = (V(t,\theta,\mathbf{x}), W(t,\theta,\mathbf{x}), \mathbf{A}(t,\theta,\mathbf{x}))$ which components transform like a five-gradient

$$\begin{aligned} V'(t',\theta',\mathbf{x}') &= V(t,\theta,\mathbf{x}) - \mathbf{v}\cdot\mathcal{R}\mathbf{A}(t,\theta,\mathbf{x}) \\ &\quad + \frac{1}{2}\mathbf{v}^2 W(t,\theta,\mathbf{x}), \\ W'(t',\theta',\mathbf{x}') &= W(t,\theta,\mathbf{x}), \\ \mathbf{A}'(t',\theta',\mathbf{x}') &= \mathcal{R}\mathbf{A}(t,\theta,\mathbf{x}) - \mathbf{v}W(t,\theta,\mathbf{x}). \end{aligned} \tag{28}$$

and interact with matter *via* a minimal coupling rule. In such a way, inserting the electromagnetic interaction into the free equations (16), we get

$$\begin{aligned} &\Big[\alpha^{-2}\left(\gamma^0+\gamma_5\right)\left(\frac{\partial}{\partial t} - ig_+V(t,\theta,\mathbf{x})\right) \\ &\quad +\alpha^{-1}\left(\gamma\cdot\nabla - ig_+\gamma\cdot\mathbf{A}(t,\theta,\mathbf{x})\right) \\ &\quad -\gamma_5\left(\frac{\partial}{\partial\theta} - ig_+W(t,\theta,\mathbf{x})\right) - i\frac{m}{\hbar}\Big] \\ &\quad \times\psi_{(+)}(t,\theta,\mathbf{x}) = 0, \\ &\Big[\alpha^{-2}\left(\gamma^0-\gamma_5\right)\left(\frac{\partial}{\partial t} - ig_-V(t,\theta,\mathbf{x})\right) \\ &\quad +\alpha^{-1}\left(\gamma\cdot\nabla - ig_-\gamma\cdot\mathbf{A}(t,\theta,\mathbf{x})\right) \\ &\quad +\gamma_5\left(\frac{\partial}{\partial\theta} - ig_-W(t,\theta,\mathbf{x})\right) + i\frac{m}{\hbar}\Big] \\ &\quad \times\psi_{(-)}(t,\theta,\mathbf{x}) = 0. \end{aligned} \tag{29}$$

In order to see the physical content of (29) let us recall, [7], that $W(t,\theta,\mathbf{x})$ may be gauged away independently from the choice of the Galilean reference frame because the condition $W(t,\theta,\mathbf{x}) = 0$ is invariant with respect to (28). Moreover, in our approach we are giving physical, Schroedinger-like, interpretation to the wave functions with periodical dependence on θ. This means that functions which describe physical particles belong to carrier spaces of representations labeled by m which is fixed and identified with the particle mass. If $\mathcal{A}(t,\theta,\mathbf{x})$ is an electromagnetic field with quanta of zero mass then consistency with this assumption requires that $\mathcal{A}(t,\theta,\mathbf{x})$ will not depend on θ. Altogether the above assumptions reduce (28) to

$$\begin{aligned} V'(t',\mathbf{x}') &= V(t,\mathbf{x}) - \mathbf{v}\cdot\mathcal{R}\mathbf{A}(t,\mathbf{x}), \\ \mathbf{A}'(t',\mathbf{x}') &= \mathcal{R}\mathbf{A}(t,\mathbf{x}), \end{aligned} \tag{30}$$

known as the "magnetic limit"of the transformation obeyed by the electromagnetic vector potential [21]. Equations (29) take now the form

$$\begin{aligned} &\Big[\alpha^{-2}\left(\gamma^0+\gamma_5\right)\left(\frac{\partial}{\partial t} - ig_+V(t,\mathbf{x})\right) \\ &\quad +\alpha^{-1}\left(\gamma\cdot\nabla - ig_+\gamma\cdot\mathbf{A}(t,\mathbf{x})\right) \\ &\quad +i\gamma_5\frac{m}{\hbar} - i\frac{m}{\hbar}\Big]\psi_{(+)}(t,\mathbf{x}) = 0, \\ &\Big[\alpha^{-2}\left(\gamma^0-\gamma_5\right)\left(\frac{\partial}{\partial t} - ig_-V(t,\mathbf{x})\right) \\ &\quad +\alpha^{-1}\left(\gamma\cdot\nabla - ig_-\gamma\cdot\mathbf{A}(t,\mathbf{x})\right) \\ &\quad -i\gamma_5\frac{m}{\hbar} + i\frac{m}{\hbar}\Big]\psi_{(-)}(t,\mathbf{x}) = 0. \end{aligned} \tag{31}$$

For dynamical components, $\phi^2_+(t,\mathbf{x})$ and $\phi^1_-(t,\mathbf{x})$, we arrive at

$$\begin{aligned} &\Big[\left(\frac{\partial}{\partial t} - \frac{m\alpha^2}{\hbar} - ig_+V(t,\mathbf{x})\right) \\ &\quad +\frac{\hbar}{2m}\left(\nabla - ig_+\mathbf{A}(t,\mathbf{x})\right)^2 \\ &\quad +\frac{\hbar}{2m}g_+\sigma\cdot\mathrm{rot}\mathbf{A}(t,\mathbf{x})\Big]\ \phi^2_+(t,\mathbf{x}) = 0, \\ &\Big[\left(\frac{\partial}{\partial t} + \frac{m\alpha^2}{\hbar} + ig_-V(t,\mathbf{x})\right) \\ &\quad -\frac{\hbar}{2m}\left(\nabla + ig_-\mathbf{A}(t,\mathbf{x})\right)^2 \\ &\quad -\frac{\hbar}{2m}g_-\sigma\cdot\mathrm{rot}\mathbf{A}(t,\mathbf{x})\Big]\ \phi^1_-(t,\mathbf{x}) = 0, \end{aligned} \tag{32}$$

i.e., at the Pauli equations with positive and negative masses and rest energies. Last terms in both equations mean that they describe particles and antiparticles with intrinsic magnetic moments $\mu_\pm = \pm\frac{g_\pm}{2m}$. As we deal with spin onehalf particles their gyromagnetic ratio is $g_\pm/m$ and our nonrelativistic model gives the correct value for the intrinsic magnetic moment of a spin onehalf particle. This result is usually associated with the relativistic Dirac equation and treated as its consequence, but in fact it emerges in the framework of Galilean approach and it has been known from the investigations of the first order analogue of the Schroedinger equation [11].

3.4. Discrete Symmetries of Galilean Covariant Dirac Equations

Now let us introduce a charge conjugation transformation C as an antilinear operation $\psi^C_{(\cdot)} = C\psi^*_{(\cdot)}$ such that $\psi^C_{(\cdot)}$ is required to satisfy the initial equations with $g_{(\cdot)} \rightarrow -g_{(\cdot)}$. For each of the equations (29) treated separately we find out that such a transformation does not exist. However it does exist when we allow $\psi^C_{(+)} = \gamma^0\gamma^2\psi^*_{(+)}$ to obey the equation for $\psi_{(-)}$ with $g_+ = -g_-$, and *vice versa*. This is in complete agreement with the standard field theoretical interpretation of the charge conjugation transformation as a passage from particles to antiparticles. We conclude therefore that really both our equations are needed for a consistent and complete description of a physical situation.

Discussion of the charge conjugation transformation C must be completed with a discussion of two other discrete symmetries considered in field theory — the space reflection P and the time reversal T. We will illustrate it also on the examples of equations (29), *i.e.* the Galilean Dirac field interacting with Galilean electromagnetic fields.

The space reflection P is to be searched as a linear transformation $\psi^P_{(\cdot)}(t,\theta,\mathbf{x}) = P\psi_{(\cdot)}(t,\theta,-\mathbf{x})$ such that $\psi^P_{(\cdot)}$ is a solution of the equation in which

$$\begin{aligned} V(t,\theta,\mathbf{x}) &\rightarrow V(t,\theta,-\mathbf{x}), \\ W(t,\theta,\mathbf{x}) &\rightarrow W(t,\theta,-\mathbf{x}), \\ \mathbf{A}(t,\theta,\mathbf{x}) &\rightarrow -\mathbf{A}(t,\theta,-\mathbf{x}). \end{aligned} \tag{33}$$

Exactly like it was for the case of the charge conjugation C such a transformation does not exist unless we will allow $\psi^P_{(\cdot)}(t,\theta,\mathbf{x}) = \gamma^0\psi_{(\cdot)}(t,\theta,-\mathbf{x})$ to be solutions of equations in which $\psi_{(+)}$ and $\psi_{(-)}$ are interchanged.

The same mechanism occurs when the time reversal T is considered. This antiunitary transformation $\psi^T_{(\cdot)}(t,\theta,\mathbf{x}) = T\psi^*_{(\cdot)}(-t,-\theta,\mathbf{x})$ leaves (29) unchanged for

$$\begin{aligned} V(t,\theta,\mathbf{x}) &\rightarrow V(-t,-\theta,\mathbf{x}), \\ W(t,\theta,\mathbf{x}) &\rightarrow W(-t,-\theta,-\mathbf{x}), \\ \mathbf{A}(t,\theta,\mathbf{x}) &\rightarrow -\mathbf{A}(-t,-\theta,\mathbf{x}). \end{aligned} \tag{34}$$

if $\psi^T_{(\cdot)}(t,\theta,\mathbf{x}) = \gamma^1\gamma^3\psi_{(\cdot)}(-t,-\theta,\mathbf{x})$ again will satisfy the interchanged equations. We can not consider these results as unexpected — in fact they realize the situation which takes place in relativistic case. All discrete symmetries of Dirac equation change upper and lower components of the Dirac's bispinor, *i.e.* components which are connected with the description of particles and antiparticles. As they both satisfy the same equation there is no question about the equation which the transformed wave functions should satisfy. The Galilean version of the Dirac equation opens such a possibility and we have seen that it remains invariant as a whole system.

4. Conclusions

Description of physical laws in the five dimensional model of spacetime with symmetry given by the extended Galilei group has advantages in comparison with the description formulated in the usual four dimensional spacetime. The fact that we have been able to avoid projective representations not only simplifies the mathematics but also allows to apply methods elaborated in the framework of relativistic field theory and with clear geometrical meaning. We have shown here that such a geometrical approach exhibits properties of physical systems usually associated with the relativistic description. As a matter of fact they are a consequence of the requirement of symmetry of physical laws with respect to the relativity group – no matter whether it is Lorentzian or Galilean. The choice of a relativity group introduces only quantitative but not qualitative differences. One should not think about particle-antiparticle duality or the one-half spin of the electron as a consequence of the particular choice of spacetime symmetry. Such fundamental properties, being crucial for our understanding of physical world, should result from fundamental principles — in our convinction from the principle of relativity — which mathematically may be realized in different ways, dependent on physical situation. In this context we stress once again the greatness of Dirac's intuition and the power of his approach, leading (independently from mathematical details) to the discovery of physical results of fundamental meaning.

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