PACS №: 04.50.h

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Maxwell-Nordtröm Equation with an Imaginary Fifth Coordinate

Contents

Abstract

A five-dimensional field tensor with components determined by the derivatives with respect to space, time and to the fifth coordinate, of the electromagnetic four-potential and of a scalar non-electromagnetic potential follows from a five-dimensional extension of the Dirac equation. With this tensor as a basis, the Maxwell-Nordström equations are derived. It is shown that when the fifth coordinate is identified with an imaginary time-coordinate, the Maxwell-Nordström equations describe electromagnetic phenomena together with hypothetical gravitomagnetic phenomena. Plane electromagnetic waves and gravitomagnetic waves in the vacuum are discussed.

1. Introduction

The fifth dimension was introduced to physics by Nordström [1, 2] in the historically first attempt at a unification of electromagnetism with gravitation. Nordström considered the six-vector of electromagnetic field and the four-vector of gravitational field. He introduced the notion of the antisymmetric field tensor in $(4+1)-$dimensional pseudo-orthogonal space; the fifth dimension was a spatial dimension. He did not discuss the covariance of that tensor under the five-dimensional rotations. Nordström derived Maxwell's equations with supplementary terms, which were connected with the four-vector of the gravitational field, together with three additional equations, again connected with the four-vector of the gravitational field. The supplementary terms in Maxwell's equations as well as the additional three equations for the first time appeared in his theory. Once the field equations unifying electromagnetism and gravitation were derived, Nordström assumed that the derivatives with respect to the real fifth coordinate vanish. With this assumption he removed any coupling between the electromagnetic and the gravitational fields. That coupling depends on the first derivatives of the fields with respect to the fifth coordinate. The Maxwell-Nordström equations [1] were rederived in [3]. That derivation represents a generalization to five dimensions of Sommerfeld's [4] derivation of the tensor form of Maxwell's equations. It holds for a spatial as well as a temporal character of the fifth coordinate. It was pointed out in [3], however, that if the invariance under time reversal of the Maxwell-Nordström equations is required, the fifth coordinate must be a time coordinate.

The notion of a second time variable is not alien to physics. We especially refer to the papers of Horwitz and his coworkers [5–8], in which a five-dimensional theory of electromagnetism was presented. In this theory the system develops on the four-dimensional space-time manifold $(\vec{x}, t)$, according to an evolution parameter, "universal time"τ. The signatures $(4, 1)$ and $(3, 2)$ of the five-dimensional metric are considered, and the reasons for preference of the metric with signature $(3, 2)$ are explained. The five-dimensional field equations derived in [6] are called "pre-Maxwell"equations, and the respective fields which obey those equations are called "pre-Maxwell"fields. The "pre-Maxwell"equations, when referred to the Cartesian system of axes, formally represent a counterpart of Maxwell-Nordström equations (see Section 4 in [6]), with imaginary fifth coordinate, although Nordström's paper is not quoted in [6]. Maxwell's equations are obtained in [6] from the "pre-Maxwell"equations after integration over the evolution parameter τ. In Nordström's paper [1], Maxwell's equations are obtained from his five-dimensional equations by fixing the value of the spatial fifth coordinate. A possible physical meaning of the evolution parameter τ was not discussed in [6–8].

We observe that the de Sitter group $SO(4, 1)$ is at the basis of the paper of Kadyshevsky [9] where the role of the *fundamental length* in quantum field theory is discussed. The ideas of that paper are developed in the papers of Kadyshevsky and his coworkers [10–12], where the fifth dimension serves in a formulation of quantum field theory containing a *fundamental mass* or a *fundamental length.*

In general relativity a second time variable was introduced and discussed in the quality of a universal parametric "historical time"by Horwitz and Piron [5], or as its generalization by Burakovsky and Horwitz [13]. In the Kaluza-Klein theory three objections were formulated against the timelike signature of the fifth coordinate [2, 14, 15]:

(1) When in the five-dimensional action the integration over the fifth coordinate is performed, provided that all derivatives with respect to the fifth coordinate are omitted and the cylinder condition is accepted, the Maxwell action comes out with an opposite sign to that of the Einstein action. This is considered to be incorrect.

(2) The existence of tachyons follows from the accepted cylinder condition.

(3) There would appear closed time curves. The problem of closed time curves in four-dimensional gravity was investigated in a series of papers of Friedman, Thorne and their coworkers [16–18].

A particular attention was paid to the question whether closed time curves violate the causality principle. The available answer does not seem to be conclusive in that respect. In the case of the five-dimensional gravity with two time coordinates, an analogous investigation has not been undertaken. It is an open question whether the second objection concerning the existence of tachyons is relevant for noncompactified Kaluza-Klein theories. As to the first objection, it seems that the relative sign of Einstein and Maxwell actions should be the outcome of a five- or higher-dimensional relativity theory, and that an answer to the question which relative sign is correct is not yet finally settled. The attitude towards a timelike signature of the fifth coordinate is less restrained in the book of Wesson [19], in which a spacelike or a timelike signature of the fifth coordinate is admitted, depending on the physical problem in question.

The idea of two times in a physical theory is a leitmotiv in the investigations of I. Bars and his coworkers [20–22]. They have shown that two-time physics provides a new perspective for the understanding of the one-time dynamics, from a higher-dimensional point of view; from a single action formula of two-time physics, with the application of gauge theory, diverse one-time dynamical systems can be obtained.

In the present paper, the second time variable is less a theoretical tool applied for unifying a theory, but rather a physical parameter which conditions the appearance of new phenomena.

We observe that the second time variable was employed in a noncompactified Kaluza-Klein type theory, in the determination of a new form of Schwarzschild solution, depending on mass and electric charge [24].

It will appear that Maxwell-Nordström equations can be derived from the field tensor implied by a five-dimensional form of the Dirac equation. This derivation is independent of a spatial or a temporal character ascribed to the fifth coordinate. The consequences which can be drawn from those equations, however, do depend on the character of the fifth coordinate. The five-dimensional form of the Dirac equation which will be considered, yields a physical basis for ascribing time character to the fifth coordinate. We will summarize the main points of the respective argument.

We observe that Dirac's paper [23] on the electron wave equation in de Sitter space contains an alternative form of his equation in Minkowski space, namely

$$[\gamma_\alpha(\partial_\alpha - i a_\alpha) - i\gamma_5\kappa]\Psi = 0 \tag{1}$$

with $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and $a_\alpha = (e/\hbar c)A_\alpha, \alpha = 1, ..., 4$, with e denoting the electron charge and A_α denoting the components of the four-potential, where x_1, x_2, x_3 are real and $x_4 = ict$, with c denoting the speed of light in the vacuum, and $\kappa = mc/\hbar$ with m denoting the electron rest mass. This equation appears by Dirac [23] for the free-electron case as the unnumbered equation at the bottom of p. 663, and for an electron in an external field it follows from his equation (33) on p. 664.

On the basis of Eq. (1), a five-dimensional form of the Dirac equation, which is covariant with respect to pseudo-orthogonal rotations belonging to the $SO(4,1)$ or $SO(3,2)$ groups, for real or imaginary fifth coordinate, was introduced in [25,26], respectively, in the form analogous to Eq. (1),

$$[\gamma_\mu(\partial_\mu - ia_\mu) - i\gamma_5\kappa]\Psi = 0 \qquad (2)$$

where $a_\mu, \mu = 1,...,4$ are those in Eq. (1), and $a_5 = m\chi/\hbar c$ or $im\chi/\hbar c$, for x_5 real or imaginary, respectively, with χ denoting a real, non-electromagnetic scalar potential.

The alternative form of the Dirac equation in Eq. (1) as well as its five-dimensional form in Eq. (2) are not invariant under the operation of space inversion [25]. This is due to the presence of γ_5 in these equations. The covariance groups of the five-dimensional equation in Eq. (2) therefore are $SO(4,1)$ or $SO(3,2)$, depending on real or imaginary fifth coordinate, respectively.

Acting on Eq. (2) from the left with the operator $(\gamma_\nu D_\nu - i\gamma_5\kappa)$, with $D_\nu = (\partial_\nu - ia_\nu)$, we obtain the equation

$$[\gamma_\nu\gamma_\mu D_\nu D_\mu - 2i\kappa(\partial_5 - ia_5) - \kappa^2]\Psi = 0. \qquad (3)$$

The first term in the square bracket, for $\mu = \nu$, together with the term $-\kappa^2$, yields a five-dimensional form of the Klein-Gordon equation. For $\mu \neq \nu$, the first term in the square bracket of Eq. (3) yields the expressions

$$-i\gamma_\mu\gamma_\nu(\partial_\mu a_\nu - \partial_\nu a_\mu)\Psi. \qquad (4)$$

The term in the brackets, multiplied by $\hbar c/e$ is a component of the five-dimensional rotation of the five-potential $A_\mu = (\hbar c/e)\, a_\mu$, $\mu = 1,...,5$, appearing in Eq. (2). The five-dimensional field tensor, determined by the five-dimensional rotation of the five-potential A_μ, will play a basic role in the present derivation of the Maxwell-Nordström equations. In the Minkowski subspace, we obtain from Eq. (4) the components of the electromagnetic field tensor, which were calculated in an analogous way in [27]. The remaining terms appearing in the iterated equation (3) were discussed in [25], however, they need not be considered at this place.

The space or time character of the fifth coordinate is of primary importance for the physical contents ot the Maxwell-Nordström equations. We therefore will summarize the results of the discussion in [25] concerning the properties of the fifth coordinate x_5, appearing in the five-dimensional form of the Dirac equation in Eq. (2) and in the field tensor in Eq. (4).

On the assumption that the fifth coordinate is affected by the operations of charge conjugation C, space inversion P and time reversal T, the conditions of invariance of that equation under the combined CPT transformations were determined in [25]. It was demonstrated that the fifth coordinate x_5 has to be odd with respect to the operation of charge conjugation. Under this condition the invariance under CPT transformations takes place when either (1) the fifth coordinate is even under $P-$operation and odd under $T-$operation, or (2) the fifth coordinate is odd under $P-$operation and even under T-operation. It was shown that in both cases the odd character of x_5 under charge conjugation is indispensible for the invariance of the five-dimensional Dirac equation in Eq. (2) under CPT transformations.

A choice between these two possibilities was made in [25], while considering the behaviour of the five-dimensional field tensor determined in Eq. (4). It has been shown in [25], and it can be checked by examining Eqs. (8) and (9) in Sec. 2, that if x_5 is even under space inversion and odd under time reversal (case (1)), then all components of the five-dimensional field tensor exhibit a definite behaviour under $P-$ and $T-$operation: they either change their sign or not. If x_5 is odd under $P-$operation and even under $T-$operation (case (2)), the components of the field tensor, which are connected with the fifth dimension, do not exhibit a definite character under $P-$ and $T-$operation : under each of these operations, in these components of that tensor, one term changes sign and the other term does not.

On the basis of this result the choice of option (1) was made in [25], that the fifth coordinate is odd under charge conjugation and odd under time reversal, while it is even under space inversion.

It can be verified that the above two possibilities of behaviour of the coordinate x_5 under CPT operations which in [25] were established for a real fifth coordinate having time character, carry over on the case of imaginary x_5. The above specified result concerning the behaviour of the field tensor under $P-$ and $T-$operation also holds for imaginary x_5.

When choice (1) is made, we write $x_5 = icu$, where c denotes the speed of light in the vacuum, and interprete u as the second time coordinate. We have to remember, however, that the second time coordinate u differs from the time coordinate t with respect to the operation of charge conjugation, since it is odd under charge conjugation.

At this point the question may be raised for the legitimacy of connecting the speed of light c with the second time coordinate. We could write $x_5 = ic'u$, with $c' = ac, a = const$. The constant a then would appear in the terms connected with x_5, without affecting the form of the equations. We will not discuss this possibility.

It will appear that Maxwell-Nordström equations can be derived starting from the five-dimensional field tensor determined in Eq. (4). If the fifth coordinate is real and space character is ascribed to it, we obtain the equations in Nordström's paper [1]. If time character is ascribed to the imaginary fifth coordinate, the Maxwell-Nordström equations imply the appearence of a new type of radiation which, owing to the fields which constitute it, may be called a gravitomagnetic radiation.

2. The Five-Dimensional Field Tensor

We consider the case when x_5 is an imaginary time coordinate, with the properties (1), specified in Section 1. This means that the coordinate $x_5 = icu$ is odd under charge conjugation and odd under time reversal, and is even under space inversion. In Eq. (2) we have the five-potential

$$\begin{aligned}\vec{\mathcal{A}} &= (A_1, A_2, A_3, A_4, A_5) \\ &= \left(A_x, A_y, A_z, \frac{i}{c}\phi, \frac{i}{c}\frac{m}{e}\chi\right). \end{aligned} \quad (5)$$

This consists of the three real components A_x, A_y, A_z, referred to a Cartesian system of coordinates (x_1, x_2, x_3) and of two imaginary components, proportional respectively to a scalar electric potential ϕ and a scalar non-electromagnetic potential χ. Under rotations in $(3+2)-$dimensional pseudo-orthogonal space, the components of the five-potential transform like the components of the five-vector $(x_1, x_2, x_3, x_4, x_5)$. We have changed the units in Eq. (5) in comparison with Eqs.(1) and (2), introducing the factor $1/c$ to the A_4 and A_5 components. In this way we will work with the units accepted in [4]. This allows for a direct comparison of the here derived formulae with those in the Minkowski subspace given in [4]. From Eqs. (4) and (5) we obtain the components of the field tensor in the form

$$\partial_j A_k - \partial_k A_j = B_i, \qquad i,j,k = 1,2,3 \quad (6)$$

$$\partial_j A_4 - \partial_4 A_j = -\frac{i}{c}E_j, \qquad j = 1,2,3 \quad (7)$$

$$\partial_j A_5 - \partial_5 A_j = -\frac{i}{c}G_j, \qquad j = 1,2,3 \quad (8)$$

$$\partial_4 A_5 - \partial_5 A_4 = -\frac{1}{c}G_0 \quad (9)$$

where B_i and E_j are the components of the magnetic induction and of the electric field, respectively, and G_j and G_0 are new fields which have to be identified.

We notice that on the condition that the vector potential is a function of both time variables $\vec{A} = \vec{A}(\vec{r}, t, u)$, where $\vec{r}$ denotes the position vector, the field G_j is determined by an expression analogous to the electric field. Firstly, the scalar electric potential A_4 has its counterpart in the scalar non-electromagnetic potential A_5. Secondly, we are dealing with time-derivatives of the vector potential $\vec{A}$, with respect to time t in the case of electric field, and with respect to time u in the case of the field G_j. On the basis of this analogy, we can assume that the field G_j is proportional to a hypothetical gravitational field. The field G_j in Eq. (8) has the same dimension as the electric field in Eq. (7), hence, for dimensional reasons, G_j itself cannot be the gravitational field but it can be proportional to the gravitational field. The respective proportionality coefficient then has the dimension of *mass/electric charge*. A coefficient with this dimension and having microscopic meaning, in the situation when the components of the field tensor are connected with the Dirac equation, is the ratio of electron mass and charge m/e. We therefore will assume that

$$\vec{G} = \frac{m}{e}\vec{G}' \quad (10)$$

with $\vec{G}'$ denoting a hypothetical gravitational field. In the following formulae, to avoid writing the factor m/e, we will use the field $\vec{G}$, remembering that the gravitational field $\vec{G}' = (e/m)\vec{G}$.

It is seen that once the imaginary fifth coordinate x_5 is related with the second time coordinate, there arises the possibility of identifying the field $\vec{G}'$ with a gravitational field in $(3+2)-$dimensional flat space. In the five-dimensional field tensor, this field appears as a gravitational counterpart of an electric field E_j.

We observe that the identification of the field G'_j with a gravitational field is also possible for a real x_5, if it is related with the second time coordinate. Some consequences of a real time coordinate $x_5 = cu$ were discussed in [3,25].

A tentative suggestion is that the field G_0, which is independent of the vector potential $\vec{A}$, and depends only on the derivatives of the two scalar potentials A_4 and A_5, relative to times t and u, respectively, is, with the accuracy to a constant coefficient, an analogue of the Brans-Dicke scalar field [2,15,19]. The field G_0 in Eq. (9) has the same dimension as the electric field in Eq. (7). If the Brans-Dicke scalar field couples with mass, the proportionality constant between the field G_0 in Eq. (9) and the field G'_0 here identified with the Brans-Dicke field again has the dimension of *mass/electric charge* and hence

$$G_0 = \frac{m}{e}G'_0. \quad (11)$$

To avoid writing the factor m/e, we will write the following formulae in terms of the field G_0.

3. The Maxwell-Nordström Equations

The following calculation represents an extension to five dimensions of Sommerfeld's calculation [4] of the tensor form of Maxwell's equations. The field ten-vector now is defined by

$$F = c\,\mathrm{Curl}\,\vec{\mathcal{A}} = (c\vec{B},\ -i\vec{E},\ -i\vec{G},\ -G_0) \quad (12)$$

where the fields $\vec{B}$, $\vec{E}$, $\vec{G}$ and G_0 were defined in Eqs. (5) through (9). This expression is an extension of the six-vector which appears in [4] in the Minkowski space to a ten-vector in (3+2)-dimensional space, with coordinates $(x_1, x_2, x_3, x_4 = ict, x_5 = icu)$. The symbol "Curl" denotes a five-dimensional rotation operation. The components of Curl $\vec{\mathcal{A}}$ are defined in Eqs. (6) through (9).

The antisymmetric field tensor connected with this ten-vector has the form

$$F = \begin{bmatrix} 0 & cB_z & -cB_y & -iE_x & -iG_x \\ -cB_z & 0 & cB_x & -iE_y & -iG_y \\ cB_y & -cB_x & 0 & -iE_z & -iG_z \\ iE_x & iE_y & iE_z & 0 & -G_0 \\ iG_x & iG_y & iG_z & G_0 & 0 \end{bmatrix}. \tag{13}$$

We next define the excitation ten-vector, which represents an extension to five dimensions of the respective excitation six-vector in the Minkowski space given in [4],

$$\begin{aligned} f &= \sqrt{\varepsilon_0/\mu_0}\, c\, \mathrm{Curl}\, \vec{\mathcal{A}} \\ &= (\vec{H},\, -ic\vec{D},\, -i\varepsilon_0\, c\vec{G},\, -\varepsilon_0\, c\, G_0). \end{aligned} \tag{14}$$

The respective five-dimensional antisymmetric second-rank tensor has the form

$$f = \begin{bmatrix} 0 & H_z & -H_y & -icD_x & -i\varepsilon_0 cG_x \\ -H_z & 0 & H_x & -icD_y & -i\varepsilon_0 cG_y \\ H_y & -H_x & 0 & -icD_z & -i\varepsilon_0 cG_z \\ -icD_x & icD_y & -icD_z & 0 & -\varepsilon_0 cG_0 \\ -\varepsilon_0 icG_x & -\varepsilon_0 icG_y & -\varepsilon_0 icG_z & \varepsilon_0 cG_0 & 0 \end{bmatrix}. \tag{15}$$

In the Minkowski subspace, the expressions for the tensors F and f reduce to those given in [4]. We now define the current density five-vector $\vec{\Gamma}$ in the form

$$\begin{aligned} &\Gamma_1 = j_x, \quad \Gamma_2 = j_y, \quad \Gamma_3 = j_z, \\ &\Gamma_4 = ic\rho_e, \quad \Gamma_5 = i\frac{ec}{m}\rho_m \end{aligned} \tag{16}$$

with ρ_e denoting the electric charge density. In comparison with that in [1, 3], the fifth component Γ_5 now is positive and imaginary, and the density of gravitational mass ρ_g which appears in [1, 3] now is replaced by the density of the total electron mass ρ_m.

The force-density five-vector $k_n, n = 1, ..., 5$, is defined in analogy with the force-density four-vector in [4] by the expression,

$$k_n = \frac{1}{c}\sum_{m=1}^{5} \Gamma_m\, F_{nm}, \qquad n = 1, ..., 5 \tag{17}$$

with Γ_m and F_{nm} given in Eqs. (16) and (13), respectively. From this expression it follows that the hypothetical gravitational field $\vec{G}'$, and the field G_0', here identified with the Brans-Dicke field, are coupled with mass density.

Three of the Maxwell-Nordström equations are obtained by performing the five-dimensional tensor-divergence operation Div on the tensor f in Eq. (15) and equating the respective terms to the components of the current density five-vector $\vec{\Gamma}$ in Eq. (16)

$$(\mathrm{Div} f)_j = \sum_{k=1}^{5} \frac{\partial f_{jk}}{\partial x_k} = \Gamma_j, \qquad j = 1, ..., 5. \tag{18}$$

Remembering that $x_5 = icu$, we obtain for $j = 1, 2, 3$ the equation

$$\mathrm{curl}\, \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} + \varepsilon_0 \frac{\partial \vec{G}}{\partial u} \tag{19}$$

with $\vec{j} = (j_x, j_y, j_z)$ in Eq. (16). For $j = 4$ we obtain

$$\mathrm{div}\, \vec{D} + \frac{\varepsilon_0}{c}\frac{\partial G_0}{\partial u} = \rho_e \tag{20}$$

and for $j = 5$ we obtain

$$\mathrm{div}\, \vec{G} - \frac{1}{c}\frac{\partial G_0}{\partial t} = \frac{e}{m\varepsilon_0}\rho_m. \tag{21}$$

We notice that when Eqs. (9) and (10) are considered, from Eq. (21) it follows that the mass density ρ_m is the source of the gravitational field $\vec{G}'$.

The remaining Maxwell-Nordström equations are calculated from the dual tensor of the field tensor F in Eq. (13). The components of this dual tensor are defined by

$$F^*_{\alpha\beta\gamma} = \frac{1}{5!}\varepsilon_{\alpha\beta\gamma\mu\nu} F_{\mu\nu} \tag{22}$$

where the star $*$ denotes the dual tensor, and $\varepsilon_{\alpha\beta\gamma\mu\nu}$ is the completely antisymmetric tensor, with $\varepsilon_{12345} = 1$, (for example, see [28]). The elements of the three-dimensional matrix F^* connected with that dual tensor are conveniently represented with the help of five two-dimensional matrices, which correspond to fixed values of the index α of the dual tensor in Eq. (21). It can be verified that the dual tensor of the tensor F is antisymmetric in all pairs of indices. Contracting the dual tensor by means of the five-dimensional tensor-divergence with respect to its third index and equating the contracted tensor to zero, we obtain the equations

$$\sum_{\gamma=1}^{5} \frac{\partial F^*_{\alpha\beta\gamma}}{\partial x_\gamma} = 0, \qquad \alpha, \beta = 1, ..., 5. \tag{23}$$

Inserting into this equation the five two-dimensional matrices representing $F^*_{\alpha\beta\gamma}$, we obtain twenty equations of which ten are different (each equation appears twice). In the Cartesian coordinates they have the form

$$\operatorname{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{24}$$

$$\operatorname{div}\vec{B} = 0, \tag{25}$$

$$\operatorname{curl}\vec{G} = -\frac{\partial \vec{B}}{\partial u}, \tag{26}$$

$$\frac{\partial \vec{E}}{\partial u} - \frac{\partial \vec{G}}{\partial t} + c\operatorname{grad} G_0 = 0. \tag{27}$$

where again the fields $\vec{G}$ and G_0 are related with the gravitational and Brans-Dicke field, by Eqs. (9) and (10), respectively.

We observe that the Maxwell-Nordström equations written in the form of Eqs. (18) and (23) are covariant under the operations of the $SO(3,2)$ group.

It is seen that the relations of the fields $\vec{E}$ and $\vec{G}$ (or $\vec{G}'$) with the magnetic field $\vec{B}$ in Eqs. (24) and (26), respectively, are formally the same. The only difference appears in the times t and u, which appear in the time derivatives. A similar type of correspondence appears in Eq. (19). The conclusion therefore can be drawn that the Maxwell-Nordström equations with an imaginary fifth coordinate having a second-time character, describe electromagnetic together with gravitomagnetic phenomena.

The gravitomagnetic field is known as a fundamental weak-field prediction of Einstein's theory of general relativity [29]. Gravitomagnetic field in general relativity is generated by currents of mass and is formally analogous to a magnetic field generated by currents of electric charge.

In the present case the hypothetical gravitomagnetic field is connected with the fifth dimension to which time character is ascribed. There appears the question for the vector source of this gravitomagnetic field. If the vector source were a mass current, this would involve, as it seems, a parametrization of electron motion with respect to two times, t and u. This would match the two times in the five-dimensional form of the Dirac equation in Eq. (2).

We finally notice that in the Cartesian coordinates, introducing the magnetic field intensity vector $\vec{H} = \mu_0^{-1}\operatorname{curl}\vec{A}$ into Eq.(19), we obtain

$$\triangle\vec{A} - \frac{1}{c^2}\Big(\frac{\partial^2\vec{A}}{\partial t^2} + \frac{\partial^2\vec{A}}{\partial u^2}\Big) = -\mu_0\,\vec{j} \tag{28}$$

on the condition that we also have

$$\sum_{\mu=1}^{5}\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{29}$$

which represents the Lorentz condition extended to five dimensions, which is covariant under the operations of the group $SO(3,2)$.

4. The Poynting Vector

The scalar product of both sides of Eq.(24) with $\vec{H}$, and of both sides of Eq.(19) with $\vec{E}$, after the subtraction of the resulting equations yields,

$$ic\vec{H}\cdot\frac{\partial\vec{B}}{\partial x_4} + ic\vec{E}\cdot\frac{\partial\vec{D}}{\partial x_4} + \vec{E}\cdot\vec{j} + c\vec{D}\cdot\frac{\partial\vec{G}}{\partial x_5} = \vec{E}\cdot\operatorname{curl}\vec{H} - \vec{H}\cdot\operatorname{curl}\vec{E} = -\operatorname{div}(\vec{E}\times\vec{H}). \tag{30}$$

The scalar product of both sides of Eq.(26) with $\vec{H}$ and of both sides of Eq.(19) with $\vec{G}$, after the subtraction of the resulting equations yields,

$$c\vec{H}\cdot\frac{\partial\vec{B}}{\partial x_5} - ic\vec{G}\cdot\frac{\partial\vec{D}}{\partial x_4} - \vec{G}\cdot\vec{j} - \varepsilon_0 c\vec{G}\cdot\frac{\partial\vec{G}}{\partial x_5} = \vec{G}\cdot\operatorname{curl}\vec{H} - \vec{H}\cdot\operatorname{curl}\vec{G} = -\operatorname{div}(\vec{G}\times\vec{H}). \tag{31}$$

The Poynting vector is determined by the right hand sides of Eqs.(30) and (31) and has the form

$$\vec{S} = \vec{E}\times\vec{H} + \vec{G}\times\vec{H}. \tag{32}$$

We observe that the Poynting vector does not contain the Brans-Dicke field G_0' defined in Eq.(11). We are dealing with a stream density of an electromagnetic field plus a stream density of a gravitomagnetic field $\vec{G}'$, defined in Eq.(10).

5. The Transversality of Electromagnetic and Gravitomagnetic Waves in the Vacuum

We now consider Eqs. (19)–(21) and (27) when $\vec{j} = 0$, $\rho_e = 0$, $\rho_m = 0$, $A_4 = 0$, and $G_0' = const$. The last condition corresponds to a fixing of the scalar field which appears in the Kaluza-Klein theory, i.e. the Brans-Dicke field, in order to obtain Einstein equations of general relativity and Maxwell equations (see, for example, [15]).

We notice that conditions $A_4 = 0$ and $G_0' = const$ imply a loss of covariance of the five-potential A_μ, and, consequently, of the Maxwell-Nordström equations with respect to the $SO(3,2)$ group of transformations. It is seen from Eq. (29) that those conditions correspond to the Coulomb gauge in four dimensions.

With the above specified conditions, from Eqs. (19),(20),(24) and (27) we obtain the equation

$$\bigtriangledown^2\vec{E} = \frac{1}{c^2}\Big(\frac{\partial^2\vec{E}}{\partial t^2} + \frac{\partial^2\vec{E}}{\partial u^2}\Big) \tag{33}$$

and from Eqs. (19), (21), (26) and (27) we obtain the equation

$$\bigtriangledown^2\vec{G} = \frac{1}{c^2}\Big(\frac{\partial^2\vec{G}}{\partial t^2} + \frac{\partial^2\vec{G}}{\partial u^2}\Big) \tag{34}$$

We now consider the $x-$component of an electric field $E_x(x,t,u)$, which propagates in the $x-$direction of a Cartesian coordinate system of axes. From Eq. (20)) we then obtain

$$\frac{\partial E_x}{\partial x} = 0 \tag{35}$$

and from Eqs. (33) and (35) we obtain

$$\frac{\partial^2 E_x}{\partial t^2} + \frac{\partial^2 E_x}{\partial u^2} = 0. \tag{36}$$

This is the equation for a harmonic function $E_x(x,t,u)$. The function $G_x(x,t,u)$ fulfils the same equation. We want the function $E_x(x,t,u)$ to be bound in the whole $(t,u)-$plane. However, a harmonic function which is bound in the whole plane is a constant function.

This shows that in a plane wave propagating in the $x-$direction, there cannot appear a periodic disturbance $E_x(x,t,u)$ or a periodic disturbance $G_x(x,t,u)$ parallel to the $x-$direction, and hence the electric field or the hypothetical gravitational field cannot have a longitudinal component in planes waves in the vacuum.

We further will consider transverse waves in which the electric field depends only on time t and the gravitational field depends only on time u. To this end it suffices to assume that the vector potential is the sum of two terms of which one depends only on time t and the other only on time u. We write

$$\vec{A}(\vec{r},t,u) = \vec{A}(\vec{r},t) + \vec{A}(\vec{r},u) \tag{37}$$

and we then obtain

$$\begin{aligned} \vec{H} &= \operatorname{curl}(\vec{A}(\vec{r},t) + \vec{A}(\vec{r},u)) \\ &= \vec{H}(\vec{r},t) + \vec{H}(\vec{r},u) \end{aligned} \tag{38}$$

and

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t}, \text{ and } \vec{G}(\vec{r},u) = -\frac{\partial \vec{A}(\vec{r},u)}{\partial u} \tag{39}$$

In this case in Eq.(19) with $\vec{j}=0$, we are dealing with terms depending only on time t and terms depending only on time u, since we obtain

$$\begin{aligned} &\operatorname{curl}(\vec{H}(\vec{r},t) + \vec{H}(\vec{r},u)) = \\ &\qquad \varepsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t} + \varepsilon_0 \frac{\partial \vec{G}(\vec{r},u)}{\partial u}. \end{aligned} \tag{40}$$

Eq. (40), then separates to two equations

$$\operatorname{curl} \vec{H}(\vec{r},t) = \varepsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t} \tag{41}$$

and

$$\operatorname{curl} \vec{H}(\vec{r},u) = \varepsilon_0 \frac{\partial \vec{G}(\vec{r},u)}{\partial u}. \tag{42}$$

These equations lead to transverse waves.

Let the wave vectors of the two types of plane waves in Eqs. (41) and (42) be $\vec{k}$ and $\vec{q}$, respectively. We will discuss two cases: **(1)** The electromagnetic $(e-m)$ plane wave and the gravitomagnetic $(g-m)$ plane wave move in the same direction, i.e. $\vec{k} \parallel \vec{q}$, and **(2)** the $(e-m)$ and $(g-m)$ plane waves move in mutually perpendicular directions, $\vec{k} \perp \vec{q}$. In both cases the polarization vectors $\vec{u}_{\vec{k}}$ and $\vec{v}_{\vec{q}}$ of the two waves will be assumed perpendicular.

(1) $\vec{k} \parallel \vec{q}$, and $\vec{u}_k \perp \vec{v}_q$. We assume that an $(e-m)$ wave moves along the $x-$axis, and is polarized in $y-$direction

$$E_y(x,t) = -\frac{\partial A_y(x,t)}{\partial t}. \tag{43}$$

Writing

$$E_y(x,t) = E_0 e^{i(k_x x - \omega t)} \tag{44}$$

we find that

$$H_z(x,t) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 e^{i(k_x x - \omega t)}. \tag{45}$$

Since, according to the asumption, the $(g-m)$ wave also moves in the $x-$direction, and its polarization vector is perpendicular to that of the $(e-m)$, the $(g-m)$ wave has to be polarized in the $z-$direction, hence we have

$$G_z(x,u) = -\frac{\partial A_z(x,u)}{\partial u}. \tag{46}$$

Writing

$$G_z(x,u) = \Gamma_0 e^{i(q_x x - \omega' u)} \tag{47}$$

where ω' denotes the frequency of the gravitomagnetic wave, we obtain for the respective magnetic field which is parallel to the $y-$direction

$$H_y(x,u) = \sqrt{\frac{\varepsilon_0}{\mu_0}} \Gamma_0 e^{i(q_x x - \omega' u)} \tag{48}$$

The magnetic field in the two waves is polarized in two mutually perpendicular directions, z and y, respectively. The ratio of the intensities of these two magnetic fields is determined by

$$\frac{|H_y(g-m)|}{|H_z(e-m)|} = \frac{|\Gamma_0|}{|E_0|} = \left|\frac{m}{e}\right| \frac{|\Gamma_0'|}{|E_0|} \tag{49}$$

where, in accordance with Eq. (11), Γ_0' denotes the amplitude of the gravitational field. Each of the two mutually perpendicular components of the magnetic field is periodic in a different time. If the interpretation of the x_5-variable as the second time-variable has a physical meaning, then in all laboratory experience we had $|H_z(t)| \gg |H_y(u)|$.

(2) $\vec{k} \perp \vec{q}$, and hence the plane waves $(e-m)$ and $(g-m)$ propagate in mutually perpendicular directions. We assumed that they are polarized in mutually perpendicular directions. We again assume that $E_y \neq 0$, and $E_x = E_z = 0$, and that the electromagnetic wave propagates in the $x-$direction. For the $(e-m)$ wave, Eqs. (44) and (45) are valid.

The direction of propagation of the $(g-m)$ wave can be parallel either to $y-$axis or to $z-$axis. We firstly assume that the gravitational field propagates in the $y-$direction, and is polarized parallel to the $x-$direction,

$$G_x(y,u) = -\frac{\partial A_x(y,u)}{\partial u} = \Gamma_0 e^{i(q_y y - \omega' u)} \qquad (50)$$

and we then obtain

$$H_z(y,u) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\Gamma_0 e^{i(q_y y - \omega' u)}. \qquad (51)$$

The magnetic field propagates in the $y-$direction and is polarized in the $z-$direction. In this case, the magnetic field in the electromagnetic wave and in the gravitomagnetic wave is polarized in the same direction

We next assume that the gravitational field propagates in the $z-$direction, and is polarized in the $y-$direction. We then have

$$G_y(z,u) = -\frac{\partial A_y(z,u)}{\partial u} = \Gamma_0 e^{i(q_z z - \omega' u)}. \qquad (52)$$

The respective magnetic field then is

$$H_x(z,u) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\Gamma_0 e^{i(q_z z - \omega' u)}. \qquad (53)$$

In this case the directions of propagation, the polarizations and the magnetic fields of the two waves are mutually perpendicular.

6. Conclusions

It was shown that the field tensor determined in Eqs. (4) or (6) through (9), which is the direct consequence of a five-dimensional form of the Dirac equation in Eq. (2), represents the basis for a derivation of the tensor form of Maxwell-Nordström equations. The derived Maxwell-Nordström equations are formally the same for a spatial [1,3] or a temporal character of the fifth coordinate, however, their physical contents is different in the two cases. In this paper a time character has been ascribed to the imaginary fifth coordinate. This second time coordinate, however, differs from the ordinary time coordinate, since it is odd under the operation of charge conjugation. This property of the second time coordinate is implied by the requirement that the components of the five-dimensional field tensor exhibit a definite behaviour (even or odd character) under space inversion and time reversal. When the fifth coordinate is identified with a second time coordinate, the Maxwell-Nordström equations describe electromagnetic phenomena together with hypothetical gravitomagnetic phenomena. In particular, gravitomagnetic waves can appear as a counterpart of electromagnetic waves.

Acknowledgement

I thank the referees for their remarks which contributed to improvements in the presentation of the results of this paper, and for drawing my attention to several important references.

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