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Dirac's Quantization of Maxwell's Theory on Non-Commutative Spaces

Contents

Abstract

Dirac's quantization of the Maxwell theory on non-commutative spaces has been considered. First class constraints were found which are the same as in classical electrodynamics. The gauge covariant quantization of the non-linear equations of electromagnetic fields on non-commutative spaces were studied. We have found the extended Hamiltonian which leads to equations of motion in the most general gauge covariant form. As a special case, the gauge fixing approach on the basis of Dirac's brackets has been investigated. The problem of the construction of the wave function and physical observables have been discussed.

1. Introduction

Quantum field theories on non-commutative (NC) spaces are usually formulated in terms of star-products of ordinary functions [1]. Non-commutative gauge theories are of interest now as they appear in the superstring theory [2]. So, in the presence of the external background magnetic field, NC coordinates can be introduced naturally [2]. The NC gauge theories can be represented as ordinary gauge theories (effective commutative theories) with the same degrees of freedom, and with the additional deformation parameter θ [3–5]. The Seiberg-Witten map between field theory on NC spaces and the corresponding commutative field theory allows us to formulate a Lagrange theory in terms of ordinary fields. The Lagrangian of the corresponding action is expanded as series of ordinary fields and a parameter that characterizes the non-commutativity. This deformation parameter of the non-commutative geometry, θ, plays the role of coupling constant of an effective Lagrangian. The parameter that characterizes non-commutativity enters the coordinate commutation relation [6,7]: $[\widehat{x}_\mu, \widehat{x}_\nu] = i\theta_{\mu\nu}$.

In the present paper, we will apply the Dirac's quantization to the Maxwell theory on non-commutative spaces. The corresponding action in terms of ordinary fields and linear in the deformation parameter has been derived in [5,8]. The one-loop corrections of the vacuum polarization of photons and BRST-shift symmetry have been considered in [9]. Some physical effects of propagation of photons in such θ-deformed Maxwell theory have been investigated in [10–12]. The energy-momentum tensor and its trace anomaly have been derived in [13].

The paper is organized as follows: In Sec. 2 the general Dirac's procedure of quantization of θ-deformed Maxwell theory is considered. The extended Hamiltonian leading to gauge covariant equations of motion is derived. The gauge fixing approach on the basis of Dirac's brackets is studied in Sec. 3. The wave function and physical observables are considered in Sec. 4. Section 5 is devoted to the discussion.

We use Lorentz-Heaviside units, and set $\hbar = c = 1$.

2. The Canonical Hamiltonian and Equations of Motion

The Maxwell Lagrangian density on NC spaces in terms of ordinary fields in the order of $\mathcal{O}(\theta^2)$ is given by [8]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{8}\theta_{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}^2 - \frac{1}{2}\theta_{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F_{\mu\nu} + \mathcal{O}(\theta^2) \quad (1)$$

where the field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2)$$

$A_\mu = (\mathbf{A}, iA_0)$ is the vector-potential of the electromagnetic field, $E_i = iF_{i4}$, $B_i = \epsilon_{ijk}F_{jk}$ ($\epsilon_{123} = 1$) are the electric field and the magnetic induction field, respectively. The Lagrangian (1) can also be represented as [13]

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)[1 + (\theta\cdot\mathbf{B})] - (\theta\cdot\mathbf{E})(\mathbf{E}\cdot\mathbf{B}) + \mathcal{O}(\theta^2) \quad (3)$$

where $\theta_i = (1/2)\epsilon_{ijk}\theta_{jk}$, $\theta_{i4} = 0$. Here we use space-like components of the tensor $\theta_{\mu\nu}$ characterizing the non-commutativity because only in this case the field theory possesses unitarity [14,15]. The terms in Eq. (3) containing the non-commutative parameter θ violate CP - symmetry. Moreover, particles in field theories on NC spaces have the dipole moments violating the CP - symmetry but CPT - symmetry remains unbroken [16,17]. It was verified that quantum electrodynamics on NC spaces at one-loop level is a renormalizable [18.8] and asymptotic free theory [19].

The Euler-Lagrange equations (field equations) follow from Eq. (1), and are given by [10,11]

$$\frac{\partial}{\partial t}\mathbf{D} - \text{rot}\mathbf{H} = 0, \qquad \text{div}\mathbf{D} = 0 \quad (4)$$

where $(\text{rot}\mathbf{H})_i = \epsilon_{ijk}\partial_j H_k$ and $\text{div}\mathbf{D} = \partial_i D_i$. The displacement ($\mathbf{D}$) and magnetic ($\mathbf{H}$) fields are defined as

$$\begin{aligned}
\mathbf{D} &= \mathbf{E} + \mathbf{d},\\
\mathbf{d} &= (\theta\cdot\mathbf{B})\mathbf{E} - (\theta\cdot\mathbf{E})\mathbf{B} - (\mathbf{E}\cdot\mathbf{B})\theta,\\
\mathbf{H} &= \mathbf{B} + \mathbf{h},\\
\mathbf{h} &= (\theta\cdot\mathbf{B})\mathbf{B} + (\theta\cdot\mathbf{E})\mathbf{E} - \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)\theta.
\end{aligned} \quad (5)$$

The second pair of equations, which is the consequence of Eq. (2), is

$$\partial_\mu \widetilde{F}_{\mu\nu} = 0, \quad (6)$$

where the dual tensor being $\widetilde{F}_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$, $\varepsilon_{\mu\nu\alpha\beta}$ is an antisymmetric tensor Levi-Civita ($\varepsilon_{1234} = -i$). Eq. (6) takes the form

$$\frac{\partial}{\partial t}\mathbf{B} + \text{rot}\,\mathbf{E} = 0, \qquad \text{div}\,\mathbf{B} = 0. \quad (7)$$

Now we apply Dirac's procedure [20] of gauge covariant quantization to the Lagrangian (1) which leads to non-linear field equations. The Lagrangian (1) is gauge invariant with the simple gauge group $U(1)$ as well as ordinary electrodynamics. There is here an infinite dimensional phase space because we deal with a field theory. As usual the Lagrangian, L, and the action, S, corresponding to the density of Maxwell's Lagrangian (1) are given by

$$L = \int d^3x\mathcal{L}, \qquad S = \int dtL \quad (8)$$

where x_i (i=1,2,3) are spatial coordinates and t is the time. We study the time evolution of fields and come, therefore, to "non-relativistic"formalism although the theory remains Lorentz covariant. The potentials, A_μ, in this formalism are "coordinates"and "velocities"are $\partial_0 A_\mu \equiv \partial A_\mu/\partial t$. According to the general formalism [20], we find from Eq. (1), with the accuracy of $\mathcal{O}(\theta^2)$, the following momenta:

$$\begin{aligned}
\pi_i &= \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} = \\
&-E_i\left[1 + (\theta\cdot\mathbf{B})\right] + (\theta\cdot\mathbf{E})B_i + (\mathbf{E}\cdot\mathbf{B})\theta_i, \quad (9)\\
\pi_0 &= \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} = 0.
\end{aligned}$$

The second equation in (9) gives a primary constraint

$$\varphi_1(x) = \pi_0, \qquad \varphi_1(x) \approx 0 \quad (10)$$

where we use Dirac's notation [20] $\approx$ for equations which hold only weakly, i.e. $\varphi_1(x)$ can have nonvanishing Poisson brackets with some variables. Eq. (10) is an infinite set of constraints for every space coordinate $\mathbf{x}$. From Eqs. (9),(5), we come to the equality $\pi_i = -D_i$, i.e. the momentum and the displacement field equal (with the opposite sign). Then, using the known Poisson bracket $\{.,.\}$ between coordinates $A_i(x)$ and momentum π_i, we arrive at

$$\{A_i(\mathbf{x},t), D_j(\mathbf{y},t)\} = -\delta_{ij}\delta(\mathbf{x}-\mathbf{y}). \quad (11)$$

From Eq. (11) it is easy to find the Poisson bracket between the magnetic induction field B_i and the displacement field D_j:

$$\{B_i(\mathbf{x},t), D_j(\mathbf{y},t)\} = \epsilon_{ijk}\partial_k\delta(\mathbf{x}-\mathbf{y}). \quad (12)$$

The same relation holds in the Born-Infeld theory [20]. In the quantized theory we have to make the substitution

$$\{B, D\} \to -i\,[B, D] \quad (13)$$

where $[B, D] = BD - DB$ is the quantum commutator. The density of the Hamiltonian found from the relation $\mathcal{H} = \pi_\mu \partial_0 A_\mu - \mathcal{L}$, with the help of Eqs. (3), (9), is given by

$$\mathcal{H} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\left[1 + (\theta \cdot \mathbf{B})\right] - (\theta \cdot \mathbf{E})(\mathbf{E} \cdot \mathbf{B}) - \pi_m \partial_m A_0. \quad (14)$$

The primary constraint (10) should be a constant of motion and, therefore, we have the condition

$$\partial_0 \pi_0 = \{\pi_0, H\} = -\partial_m \pi_m = 0 \quad (15)$$

where

$$H = \int d^3x \mathcal{H} \quad (16)$$

is the Hamiltonian. Eq. (15) guarantees that the primary constraint (10) is conserved. The secondary constraint, found from Eq. (15), is

$$\varphi_2(x) = \partial_m \pi_m, \qquad \varphi_2(x) \approx 0. \quad (17)$$

It should be noted that weak equalities, $\approx$, are not compatible with the Poisson brackets [20]. Using the equality $\pi_m = -D_m$, it is easy to see that the secondary constraint (17) is simply the Gauss law (see Eq. (4)). The time evolution of the secondary constraint

$$\partial_0 \varphi_2 = \{\varphi_2, H\} \equiv 0 \quad (18)$$

shows that there is no additional constraints. The Poisson bracket between primary, φ_1, and secondary, φ_2, constraints vanishes, $\{\varphi_1, \varphi_2\} = 0$. Thus, all constraints here are first class, and there are no second class constrains, as in classical electrodynamics [20]. The primary and secondary constraints can be considered on the same footing [20]. According to the general method [20], to acquire the total density of Hamiltonian, we add to Eq. (14) Lagrange multiplier terms $v(x)\pi_0$, $u(x)\partial_m \pi_m$, where $v(x)$ and $u(x)$ are auxiliary variables which have no physical meaning, and are connected with gauge degrees of freedom. As a result, we arrive at the total density of Hamiltonian of Maxwell's theory on NC spaces:

$$\mathcal{H}_T = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\left[1 + (\theta \cdot \mathbf{B})\right] - (\theta \cdot \mathbf{E})(\mathbf{E} \cdot \mathbf{B}) - \pi_m \partial_m A_0 + v(x)\pi_0 + u(x)\partial_m \pi_m. \quad (19)$$

The first class constraints in Eq. (19) generate gauge transformations and Eq. (19) gives the set of Hamiltonians. As the physical space is the constraint surface, we get the energy from the Hamiltonian on the constraint surface. Thus, the density energy, found from Eq. (19), is given by

$$\mathcal{E} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\left[1 + (\theta \cdot \mathbf{B})\right] - (\theta \cdot \mathbf{E})(\mathbf{E} \cdot \mathbf{B}). \quad (20)$$

The same expression, Eq. (20), was obtained in [13] from another procedure. To obtain equations of motion we have to express the density of Hamiltonian (19) in terms of fields, A_μ, and momenta, π_μ. For this, using Eq. (9), we find, with the accuracy of $\mathcal{O}(\theta^2)$, the electric field

$$E_i = -\pi_i\left[1 - (\theta \cdot \mathbf{B})\right] - (\theta \cdot \pi) B_i - (\pi \cdot \mathbf{B})\theta_i. \quad (21)$$

With the help of Eq. (21) and the equality $D_i = -\pi_i$, the total density of Hamiltonian (19) takes the form

$$\mathcal{H}_T = \frac{\pi^2 + \mathbf{B}^2}{2} + (\theta \cdot \mathbf{B})\frac{\mathbf{B}^2 - \pi^2}{2} + (\theta \cdot \pi)(\pi \cdot \mathbf{B}) + v(x)\pi_0 + (u(x) + A_0)\partial_m \pi_m \quad (22)$$

where $\mathbf{B} = \mathrm{rot}\mathbf{A}$ and fields A_i enter the density of Hamiltonian (22) in the form of the rot$\mathbf{A}$. We took into account Eq. (16) and an integration by parts to get the term $A_0 \partial_m \pi_m$ in Eq. (22). As the function $u(x)$ is arbitrary, we can make the substitution $u'(x) = u(x) + A_0$ (see [20]) in Eq. (22), so that $u'(x)$ will also be arbitrary, and the component A_0 is absorbed by the function $u(x)$. After this substitution, the A_0 does not enter the Hamiltonian and its dynamics is not defined by the Hamiltonian. This means that the component A_0 is not the physical degree of freedom. The same concerns the component π_0, which is zero (see (10)). The role of terms $v(x)\pi_0 + u(x)\partial_m \pi_m$ in Eq. (22) is to generate gauge transformations of fields which do not affect the physical state of the system.

The total density of Hamiltonian allows us to obtain the time evolution of fields. With the help of the Hamiltonian equations we find

$$\partial_0 A_i = \{A_i, H\} = \frac{\delta H}{\delta \pi_i} = \pi_i\left[1 - (\theta \cdot \mathbf{B})\right] + (\mathbf{B} \cdot \pi)\theta_i + (\pi \cdot \theta) B_i - \partial_i A_0 - \partial_i u(x), \quad (23)$$

$$\partial_0 \pi_i = \{\pi_i, H\} = -\frac{\delta H}{\delta A_i} = \partial_n\left\{\left[(\partial_n A_i) - (\partial_i A_n)\right]\left[1 + (\theta \cdot \mathbf{B})\right]\right\} + \varepsilon_{iab}\partial_b\left[\theta_a \frac{\mathbf{B}^2 - \pi^2}{2} + \pi_a(\pi \cdot \theta)\right], \quad (24)$$

$$\partial_0 A_0 = \{A_0, H\} = \frac{\delta H}{\delta \pi_0} = v(x), \qquad \partial_0 \pi_0 = \{\pi_0, H\} = -\frac{\delta H}{\delta A_0} = -\partial_m \pi_m. \quad (25)$$

Eq. (24) coincides with the first equation in (4) taking into consideration the definition (5), and Eq. (23) is nothing but the gauge covariant form of Eq. (21). The second equation in (4), Gauss's law, is the secondary constraint in the Hamiltonian formalism. As a particular case at $v(x) = \partial_0 u'(x)$ ($u'(x) = u(x) + A_0$), we arrive from Eqs. (23), (25) at the relativistic form of ordinary gauge transformations generated by constraints:

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \quad (26)$$

where $\Lambda(x) = \int dt u'(x)$. But in general case, there are here two arbitrary functions, $v(x)$, $u(x)$. Thus, the Hamiltonian equations (23),(24) give the time evolution of physical fields which are gauge-equivalent to some solutions of the Euler-Lagrange equations. In this approach, the first class constraints initiate gauge transformations and realize a gauge algebra representation. Eq. (25) represents the time evolution of non-physical fields. According to Eq. (25) the component A_0 is arbitrary function connecting with gauge degree of freedom, and π_0, $\partial_m\pi_m$ equal zero as constraints.

3. The Coulomb Gauge and Quantization of Second-Class Constraints

Let us consider the Coulomb (radiation) gauge constraints using the gauge freedom of the θ-deformed Maxwell theory. For classical electrodynamics such a procedure was considered in [21] (see also [22,23]). It should be noted that the gauge fixing procedure is beyond the Dirac's approach. The gauge degrees of freedom are present in the Dirac method. So, in this section, we consider gauge fixing approach.

With the help of the gauge freedom, described by two functions, $v(x)$, $u(x)$, or Eq. (26), we can impose new constraints as follows:

$$\varphi_3(x) = A_0 \approx 0, \qquad \varphi_4(x) = \partial_m A_m \approx 0. \tag{27}$$

The Coulomb gauge (27) does not violate the equations of motion. After fixing two components of vector-potential A_μ in accordance with Eqs. (27), the first class constraints (10), (17) become second class constraints. Indeed, the non-zero Poisson brackets of functions φ_1, Eq. (10), φ_2, Eq. (17) and φ_3, φ_4, Eq. (27), are (see [21])

$$\begin{aligned} \{\varphi_1(\mathbf{x},t), \varphi_3(\mathbf{y},t)\} &= -\delta(\mathbf{x}-\mathbf{y}), \\ \{\varphi_2(\mathbf{x},t), \varphi_4(\mathbf{y},t)\} &= \Delta_x\delta(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{28}$$

where $\Delta_x \equiv \partial^2/(\partial x_m)^2$. The pairs of "coordinates" Q_i and conjugated momenta P_i $(i = 1, 2)$ are defined as follows (see [23]):

$$Q_i = (A_0, \partial_m A_m), \quad P_i = (\pi_0, -\Delta_x^{-1}\partial_m\pi_m), \tag{29}$$

and equations

$$\{Q_i(\mathbf{x},t), P_j(\mathbf{y},t)\} = \delta_{ij}\delta(\mathbf{x}-\mathbf{y}), \tag{30}$$

$$\Delta_x^{-1} = -\frac{1}{4\pi|\mathbf{x}|}, \qquad \Delta_x\frac{1}{4\pi|\mathbf{x}|} = -\delta(\mathbf{x}) \tag{31}$$

hold. Pairs of canonical variables Q_i, P_i, Eq. (29), do not describe a true physical degrees of freedom. Therefore, in the quantum theory these operators must be eliminated. Defining the matrix of Poisson brackets as [21]

$$C_{ij} = \{\varphi_i(\mathbf{x},t), \varphi_j(\mathbf{y},t)\}, \tag{32}$$

so that the inverse matrix C_{ij}^{-1} exists [21], we may introduce the Dirac bracket [20,21]:

$$\begin{aligned} \{A(\mathbf{x},t), B(\mathbf{y},t)\}^* &= \{A(\mathbf{x},t), B(\mathbf{y},t)\} \\ &- \int d^3z d^3w \{A(\mathbf{x},t), \varphi_\alpha(\mathbf{z},t)\} \\ &\times C_{\alpha\beta}^{-1}(z,w)\{\varphi_\beta(\mathbf{w},t), B(\mathbf{y},t)\}. \end{aligned} \tag{33}$$

The inverse matrix C_{ij}^{-1} obeys the equation

$$\int d^3z C_{\alpha\gamma}(x,z)C_{\gamma\beta}^{-1}(z,y) = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}), \tag{34}$$

and is given by

$$C_{\alpha\beta}^{-1}(x,y) = \begin{pmatrix} 0 & 0 & \delta(\mathbf{x}-\mathbf{y}) & 0 \\ 0 & 0 & 0 & \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \\ -\delta(\mathbf{x}-\mathbf{y}) & 0 & 0 & 0 \\ 0 & -\dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|} & 0 & 0 \end{pmatrix}. \tag{35}$$

Using the definition of Dirac's bracket (33) and Eq. (35), and imposing the boundary condition that the fields vanish at infinity, we arrive at the same expression for the Dirac brackets as in classical electrodynamics (see [21]):

$$\begin{aligned} \{\pi_0(\mathbf{x},t), A_0(\mathbf{y},t)\}^* &= \{\pi_0(\mathbf{x},t), A_i(\mathbf{y},t)\}^* \\ &= \{\pi_i(\mathbf{x},t), A_0(\mathbf{y},t)\}^* = 0, \end{aligned} \tag{36}$$

$$\begin{aligned} \{\pi_i(\mathbf{x},t), A_j(\mathbf{y},t)\}^* &= -\delta_{ij}\delta(\mathbf{x}-\mathbf{y}) \\ &+ \frac{\partial^2}{\partial x_i \partial y_j}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \quad (i,j = 1,2,3), \end{aligned} \tag{37}$$

$$\begin{aligned} \{\pi_\mu(\mathbf{x},t), \pi_\nu(\mathbf{y},t)\}^* &= \{A_\mu(\mathbf{x},t), A_\nu(\mathbf{y},t)\}^* = 0 \\ &(\mu,\nu = 1,2,3,4). \end{aligned} \tag{38}$$

Using the well-known Fourier transformation of the Coulomb potential [24]

$$\int \frac{d^3x}{4\pi|\mathbf{x}|}e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{|\mathbf{k}|^2}, \tag{39}$$

Eq. (37) takes the form

$$\{\pi_i(\mathbf{k}), A_j(\mathbf{q})\}^* = -(2\pi)^3\delta(\mathbf{k}+\mathbf{q})\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right). \tag{40}$$

So, the projection operator in the right side of Eq. (40):

$$\Pi = (\Pi_{ij}) = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)$$

with the properties $\Pi^2 = \Pi$, $\Pi\mathbf{k} = 0$, extracts the physical transverse components of vectors.

According to the definition (33), the Dirac bracket of any operator A with a second class constraint, φ_α, vanishes, $\{A, \varphi_\alpha\}^* = 0$. Therefore, following the prescription [20], we can set all second class constraints strongly to zero. As a result, only two transverse components of the vector potential A_μ and momentum π_μ are physical independent variables. Thus, pairs of operators (29) are absent in the reduced physical phase space. Then the physical Hamiltonian of fully constrained θ-deformed Maxwell's theory becomes

$$H^{ph} = \int d^3x\mathcal{E} = \int d^3x \left\{ \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \times \left[1 + (\theta\cdot\mathbf{B})\right](\theta\cdot\mathbf{E})(\mathbf{E}\cdot\mathbf{B}) \right\}. \quad (41)$$

Equations of motion obtained from Eq. (41) are given by

$$\{\mathbf{A}, H^{ph}\}^* = \partial_0\mathbf{A} = \pi\left[1 - (\theta\cdot\mathbf{B})\right] + (\mathbf{B}\cdot\pi)\,\theta + (\pi\cdot\theta)\,\mathbf{B}, \quad (42)$$

$$\{\pi, H^{ph}\}^* = \partial_0\pi = -\mathrm{rot}\left\{\mathbf{B}\left[1 + (\theta\cdot\mathbf{B})\right] + \theta\frac{\mathbf{B}^2 - \pi^2}{2} + \pi(\pi\cdot\theta)\right\}. \quad (43)$$

In the quantum theory, with the presents of second class constraints, we have to replace Dirac's bracket by the quantum commutator according to the prescription $\{.,.\}^* \to -i[.,.]$. Only transverse components of the vector potential A_μ are physical degrees of freedom, and they remain in the theory.

4. Wave Function, Observables and Quantization of First-Class Constraints

We do not imply here the gauge conditions, described in the previous section, and we follow the direct Dirac method. The gauge degrees of freedom present as operators in a bigger linear space and constraints are operators which act on Dirac's space. One needs to extract the physical sub-space by imposing conditions.

Let us consider the problem of constructing physical states in the Hilbert space and observables. In quantized theory, the dynamical variables $\hat{A}_i$ and $\hat{\pi}_i = -\hat{D}_i$ obey here the commutation relation (see (11)):

$$[\hat{A}_i(\mathbf{x}, t), \hat{D}_j(\mathbf{y}, t)] = -i\delta_{ij}\delta(\mathbf{x} - \mathbf{y}). \quad (44)$$

The wave function $|\Psi\rangle$ satisfies the Schrödinger equation

$$i\frac{d|\Psi\rangle}{dt} = H|\Psi\rangle \quad (45)$$

where H is the Hamiltonian (16) with the density (19). In accordance with [20] the wave function (the state) must obey the following equations

$$\hat{D}_0|\Psi\rangle = 0, \qquad \partial_m\hat{D}_m|\Psi\rangle = 0, \quad (46)$$

i.e., the physical state remains unchanged, and as a result, it is invariant under the gauge transformations. In the coordinate representation the operators of the "coordinate" $\hat{A}_i$ and the momentum $\hat{\pi}_i = -\hat{D}_i$ are given by

$$\hat{A}_i\Psi[A] = A_i\Psi[A], \quad \hat{D}_\mu(x)\Psi[A] = i\frac{\delta\Psi[A]}{\delta A_\mu(x)} \quad (47)$$

where $\Psi[A]$ is the wave functional (the vector of the state). In this representation Eqs. (46) take the form [25]

$$\frac{\delta\Psi[A]}{\delta A_0(x)} = 0, \qquad \partial_i\frac{\delta\Psi[A]}{\delta A_i(x)} = 0. \quad (48)$$

Thus, constraints are restrictions on the state $|\Psi\rangle$. Both equations (46) (or (48)) are compatible if $[\hat{D}_0, \partial_m\hat{D}_m] = 0$; this is the case. So, constraints do not change the physical states $|\Psi\rangle$ being gauge invariant quantities, and are the generators of the gauge symmetry. The method of gauge fixing, described in the previous section, leads to the reduced phase space, and is equivalent to the Dirac approach.

The fields **E**, **B**, **D**, **H** are invariants of the gauge transformations and are observables (measurable quantities). As usual, real observables are represented by the Hermitian operators and must not depend on A_0.

To have normalized states and a physical interpretation to the theory, one needs to construct a scalar product of wave functions (functionals). Obviously, this product is given by the functional integral. Then, however, we should take into account the gauge degrees of freedom, and insert a gauge condition. As a result, we arrive at the necessity to introduce ghosts. Such a procedure is beyond the Dirac approach. The second way is to use the Fock basis ignoring the wave functionals. The problem of constructing the vacuum state and the scalar product on the physical state space may be realized, with the help of the Fock representation, in the same manner as in the case of classical electrodynamics (see [25]). This procedure may involve, however, the introducing negative norm states which should not appear in the physical spectrum.

5. Conclusion

We have considered quantization of the Maxwell theory on non-commutative spaces taking into consideration first class constraints as well as introducing second class constraints and the Dirac bracket. The procedure of Dirac's quantization here, on the basis of first class constraints and the Poisson bracket, is similar to the quantization of classical electrodynamics because the gauge group is the same: $U(1)$. The difference is that field equations are nonlinear in the case of the θ-deformed Maxwell theory, and, as a result, the quantization of a theory is more complicated. Dirac's method of quantization has an advantage compared to the reduced phase space approach (see [22]) that it does not violate the Lorentz invariance and locality in space.

The quantization of Maxwell's theory on non-commutative spaces within BRST-scheme, by the inclusion of the ghosts, was performed in [8,9,26].

Let me now mention some interesting phenomenon impacting on NC quantum electrodynamics (NCQED). NC theory can be verified at high energy using e^+e^- and hadronic collisions and low energy precision experiments for measurements of anomalous magnetic (AM) and electric dipole (ED) moments, as well as other CP-violating effects [27]. So, e^+e^-, $e\gamma$, and $\gamma\gamma$ cross sections within the NCQED framework depend on $\theta_{\mu\nu}$ [28]. In particular, for Möller and Bhabha scattering, cross sections depend on θ_{ij} (when i,j=1,2,3) and θ_{0j}, respectively. Note the Aharonov-Bohm effect for high energy electrons might also influence the boundary on θ [29].

There are constraints on θ parameters when splitting levels of energy in positronium and the Lamb shift [30]. Electron ED moments and muon AM moments give the boundary of $\theta < 10^{-4}$ TeV^{-1} [16,31], and $\theta < m_\mu^{-1}$ [32], respectively. In the electroweak sector, the CP violation provides the NC effect $\theta < 1$ TeV^{-1} [31,33].

NC geometry contributes some effects in cosmology [34] because in the early universe, at temperatures above $\theta^{-1/2}$ NC effects were important. That may explain the detection of high energy photons in the cosmos [35] and a global structure after the universe had expanded.

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