

A Fermion Doublet with Chiral Gauge Interaction on a Lattice

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Abstract

We present a new staggered discretization of the Dirac operator. Doubling gives only a doublet of Dirac fermions which we propose to interpret as a physical (lepton or quark) doublet. If coupled with gauge fields, an $(1 + \gamma^5)$ chiral interaction appears in a natural way. We define a generalization for curved background which does not require tetrad variables. The approach suggests a natural explanation for the three fermion families.

1. Introduction

In this paper we present a new discretization of the Dirac equation. In comparison with staggered fermions [8] it creates not four but only two flavours of Dirac fermions. This has been reached by placing not only different spin components, but also their real and imaginary parts into different nodes. These sixteen degrees of freedom (two fermions) can be understood, in some sense, as the result of "doubling" of a real scalar degree of freedom $\varphi(n)$ on the lattice.

Moreover, these two fermions live on different sublattices (ψ_o on "odd" nodes, ψ_e on "even" nodes), thus, we obtain also a single fermion (eight degrees

of freedom) on the lattice by omitting half of the lattice nodes. But, if the neutrino is a Dirac particle, all fermions appear in doublets of Dirac particles. In this context, the appearance of a fermion doublet in this discretization may be not a bug but a feature which allows to explain why Dirac particles appear in such doublets.

In our approach the complex structure is an operator among others. Moreover, there is no natural complex structure, but, instead, a quaternionic structure. The choice of a complex structure corresponds to a choice of a direction. To avoid unnecessary symmetry breaking we propose to

The natural replacement which does not depend on it – the expression $\gamma^0\gamma^1\gamma^2\gamma^3$ – we denote with ι :

$$\begin{aligned} \iota &\stackrel{def}{=} \gamma^0\gamma^1\gamma^2\gamma^3 = \beta^1\beta^2\beta^3, \\ \iota\gamma^\mu + \gamma^\mu\iota &= 0, \\ (\iota)^2 &= -1. \end{aligned} \quad (8)$$

3. Discretization of the Dirac Equation

This representation is appropriate for a discretization of the Dirac equation on a regular hyper-cubic lattice. It can be obtained in a quite simple way: We start with a naive central difference approximation

$$\partial_i\psi(n) \rightarrow \frac{1}{2a_i}(\psi(n+a_i) - \psi(n-a_i)). \quad (9)$$

This naive discretization leads to the effect of "fermion doubling". The continuous limit of this set of discrete equations gives not only the original Dirac equation, but also additional, highly oscillating components, the "doublers". In classical computations such doublers may be often ignored, but in quantum computations, where the number of degrees of freedom is important (Pauli principle) this is no longer possible. We obtain in each direction a factor to, thus, $2^4 = 16$ doublers. Fortunately, eight pairs of doublers decouple in a really simple way: It is sufficient to hold only one of the eight real components ψ^a per node. On the three-dimensional reference cube $(\varepsilon_1, \varepsilon_2, \varepsilon_3), \varepsilon_i \in \{0, 1\}$ we obtain the following locations for the eight components:

$$\begin{aligned} \psi^0 &\text{ located at } (0, 0, 0); \\ \psi^1 &\text{ located at } (1, 0, 0); \\ \psi^2 &\text{ located at } (0, 1, 0); \\ \psi^3 &\text{ located at } (1, 1, 0); \\ \psi^4 &\text{ located at } (0, 0, 1); \\ \psi^5 &\text{ located at } (1, 0, 1); \\ \psi^6 &\text{ located at } (0, 1, 1); \\ \psi^7 &\text{ located at } (1, 1, 1). \end{aligned} \quad (10)$$

What remains are sixteen degrees of freedom (eight degrees of freedom on two time degrees which we need because of our use of central differences in time) which corresponds to a doublet of Dirac fermions². Note that our discretization may be interpreted as a way to discretize the d'Alembert equation for a single scalar

²This is a variant of a well-known approach to solve the "fermion doubling" problem – so-called staggered fermions [8]. This approach reduces the doublers only by a factor four. Because we ignore the complex structure of the standard representation, we are free to place "real" and "imaginary" part of the complex fields into different nodes. This gives the additional reduction by factor two.

degree of freedom $\varphi(n)$ with central differences, which gives $2^4 = 16$ doublers.

Now, the last doublet decouples too, but in a slightly less trivial way: We can distinguish "even" and "odd" nodes on the full space-time lattice. The central difference equations on even (odd) nodes connects only values on odd (even) nodes. Thus, we obtain two fermions ψ_e and ψ_o on even resp. odd nodes. On the four-dimensional reference cube $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3), \varepsilon_i \in \{0, 1\}$ we have

$$\begin{aligned} \psi_e^0 &\text{ located at } (0, 0, 0, 0); \\ \psi_e^1 &\text{ located at } (1, 1, 0, 0); \\ \psi_e^2 &\text{ located at } (1, 0, 1, 0); \\ \psi_e^3 &\text{ located at } (0, 1, 1, 0); \\ \psi_e^4 &\text{ located at } (1, 0, 0, 1); \\ \psi_e^5 &\text{ located at } (0, 1, 0, 1); \\ \psi_e^6 &\text{ located at } (0, 0, 1, 1); \\ \psi_e^7 &\text{ located at } (1, 1, 1, 1); \end{aligned} \quad (11)$$

$$\begin{aligned} \psi_o^0 &\text{ located at } (1, 0, 0, 0); \\ \psi_o^1 &\text{ located at } (0, 1, 0, 0); \\ \psi_o^2 &\text{ located at } (0, 0, 1, 0); \\ \psi_o^3 &\text{ located at } (1, 1, 1, 0); \\ \psi_o^4 &\text{ located at } (0, 0, 0, 1); \\ \psi_o^5 &\text{ located at } (1, 1, 0, 1); \\ \psi_o^6 &\text{ located at } (1, 0, 1, 1); \\ \psi_o^7 &\text{ located at } (0, 1, 1, 1). \end{aligned} \quad (12)$$

But, instead of removing one sub-mesh to describe a single Dirac fermion, we propose to accept above doublers as a way to describe a physically meaningful flavour doublet. Remarkably, if the neutrino is a standard Dirac particle, then all fermions of the standard model appear in doublets. The appearance of a fermion doublet in our approach may be, therefore, not a bug but a feature which allows to explain the existence of these doublets.

4. Complex Structures

To connect the Dirac particle with a gauge field we need a complex structure – a multiplication with i . In our real representation this is an operator. It's basic property is $i^2 = -1$. To be compatible in the usual way with the Dirac equation, we also need $[\gamma^\mu, i] = 0$. This does not define the complex structure uniquely. We have several interesting candidates for a complex structure:

$$i = \beta^1\beta^2 = \iota\beta^3, \quad (13)$$

$$j = \beta^2\beta^3 = \iota\beta^1, \quad (14)$$

$$k = \beta^3\beta^1 = \iota\beta^2 \quad (15)$$

which together define a quaternionic structure:³

³The classical representation $ij = k$ can be obtained using reverse signs for i, j, k , but we prefer this sign convention because it gives $\gamma^5 = \beta^3$.

$$\begin{aligned}
 ij &= -ji = -k, \\
 jk &= -kj = -i, \\
 ki &= -ik = -j, \\
 i^2 &= j^2 = k^2 = -1.
 \end{aligned}
 \tag{16}$$

For each candidate for a complex structure, we obtain an own operator $\gamma^5 \stackrel{def}{=} -i\iota$. Especially for $i = \iota\beta^3$ we obtain $\gamma^5 = -\iota\beta^3\iota = \beta^3$.

4.1. Fermion Families and Lattice Distortions

Thus, it seems that to fix the complex structure we somehow have to break spatial symmetry. But there is a simple way out of this. Instead of one scalar degree of freedom $\varphi(n)$ on the lattice, we can consider a vector field – thus, three components $\varphi^i(n)$. Now, each component has a natural "preferred direction" and, therefore, a natural complex structure.

Moreover, a vector on a lattice is a quite natural degree of freedom. It is, for example, the natural way to describe lattice distortions with a shift vector field $u^i(n)$.

On the other hand, in the standard model we have three families – three copies of each fermion. This suggests to explain on the kinematic level the three fermion families using the hypothesis that the fundamental degrees of freedom of the "theory of everything" are three-dimensional vector fields $u^i(n)$. This idea becomes really meaningful only in an ether-theoretical context, where relativistic symmetry is considered as not fundamental but an effective large scale symmetry, and distortions of an "ether crystal" are natural degrees of freedom. We consider such an approach in more detail in appendix .

For our symmetry problem – the choice of the complex structure – other solutions are possible. We can use them all in expressions like

$$iA^1 + jA^2 + kA^3 \tag{17}$$

which allows to connect with an $SU(2)$ or $SO(3)$ gauge field. We can also try to connect colors with spatial directions to define the preferred complex structure differently for each colored quark. Last not least, it may be really broken (possibly the $U(1)$ part of electroweak interaction). These different solutions possibly even do not contradict each other.

But it seems important to notice that the choice of the complex structure is not natural, requires a symmetry breaking of the quaternionic structure. This point may be important for the understanding of symmetry breaking in the standard model.

5. Chiral Symmetry on the Lattice

Let's assume now that one of the complex

structures, namely $i = \iota\beta^3$, has been chosen. To understand chiral symmetry we have to define the operator $\gamma^5 = \beta^3$ on the lattice. Note that it cannot be anymore a pointwise operator as for Wilson fermions and staggered fermions – it connects components which are located in different points. Thus, the original, continuous definition is not enough to define γ^5 on the lattice. But there is one very natural choice: the shift operator in z-direction. Modulo 2, on the reference cube, it is indeed γ^5 . And, most important, it anticommutes with the Dirac operator even on the lattice:

$$\gamma^5 D + D\gamma^5 = 0. \tag{18}$$

But, different from the γ^5 in continuous theory, it's square is no longer 1, but a shift in z-direction. This is a very nice and beautiful way to break the exact chiral symmetry $(\gamma^5)^2 = 1$. Note that γ^5 exchanges the even and odd fermion:

$$\begin{aligned}
 \gamma^5 \psi_e(n) &= \psi_o(n - a_z), \\
 \gamma^5 \psi_o(n) &= \psi_e(n - a_z).
 \end{aligned}
 \tag{19}$$

This gives the subspaces $\psi_{\pm} = \frac{1}{2}(1 \pm \gamma^5)\psi$ a quite obvious physical meaning: they are the subspaces defined by $\psi_e \approx \pm\psi_o$. On the other hand, because ψ_e and ψ_o are located on different sub-meshes, we do not have a natural symmetric and exact definition of the projectors.

Now, the most beautiful surprise we find thinking about the connection to gauge fields. As usual, we describe gauge degrees of freedom as located on edges. Now, a quite natural, "naive" symmetric lattice interpolation for interaction terms $A\psi$ between fermions and the gauge field is

$$A_{n+a_i, n} \frac{\psi(n+a_i) + \psi(n)}{2}. \tag{20}$$

Thus, we obtain a dependency on terms of type $\psi(n+a_i) + \psi(n)$ in a natural way. Now, these terms are $\approx \psi(n)$ for ψ_+ but ≈ 0 for ψ_- – thus, our interaction terms with gauge fields have exactly the $(1 + \gamma^5)$ form of interaction we need in chiral gauge theory! Thus, in this discretization we have not only a possibility to describe chiral gauge fields, but they appear almost by themselves, in a quite natural way.

This effect disappears if we have several components and the gauge field connects them. Thus, our $(1 + \gamma^5)$ type interaction appears only inside the doublet. Outside the doublets we obtain a vector-like interaction – as it should be in chromodynamics.

6. The Dirac Operator on the de Rham Complex

The discretization may be generalized to the case of a general metric background in a quite simple way – as the exterior bundle or the de Rham complex. The de

Rham complex and its Dirac operator is well-known, but has the "wrong" dimension for a single fermion: in four dimension, we have $2^4 = 16$ components. Instead, fermions are usually described on a curved background using tetrad variables for the metric (see, for example, [1]).

In our approach we have to describe a doublet of Dirac fermions, and in this context these sixteen components of the exterior bundle is exactly what we need. In this sense, the "fermion doubling" problem has a natural analogon in continuous theory on a curved background: only a pair of fermions may be described on a curved metric background $g_{\mu\nu}$ without tetrad variables.

6.1. Hodge Theory

Let's remember the basic formulas for the Dirac operator in the exterior bundle (see, for example, [14]). The exterior bundle or de Rham complex $\Omega = \sum_{k=0}^n \Omega^k$ consists skew-symmetric tensor fields of type $(0, k), 0 \leq k \leq n$ which are usually written as differential forms

$$\psi = \psi_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k \quad (21)$$

The exterior bundle Ω has dimension 2^n in the n -dimensional space. The most important operation on Ω is the external derivative $d: \Omega^k \rightarrow \Omega^{k+1}$ defined by

$$(d\psi)_{i_1 \dots i_{k+1}} = \sum_{q=1}^{k+1} \frac{\partial}{\partial x^{i_q}} (-1)^q \psi_{i_1 \dots \hat{i}_q \dots i_{k+1}} \quad (22)$$

where \hat{i}_q denotes that the index i_q has been omitted. Its main property is $d^2 = 0$. In the presence of a metric, we have also the important $*$ -operator $\Omega^k \rightarrow \Omega^{n-k}$:

$$(*\psi)_{i_{k+1} \dots i_n} = \frac{1}{k!} \varepsilon_{i_1 \dots i_n} g^{i_1 j_1} \dots g^{i_k j_k} \psi_{j_1 \dots j_k} \quad (23)$$

with $*^2 = (-1)^{k(n-k)} \text{sgn}(g)$. This allows to define a global inner product by

$$(\phi, \psi) = \int \phi \wedge (*\psi) = \int \psi \wedge (*\phi). \quad (24)$$

It turns out that the adjoint operator of d^* of d is

$$d^* = (-1)^{rn+n+1} * d *. \quad (25)$$

In this general context we can define the Laplace operator as

$$\Delta = dd^* + d^*d. \quad (26)$$

Then, the Dirac operator (as its square root) can be defined as

$$D = d + d^*. \quad (27)$$

Indeed, we have $d^2 = 0$ as well as $(d^*)^2 = 0$. Sometimes the \mathbb{Z}_2 graduation is also useful: $\varepsilon\psi =$

$(-1)^k \psi$ if $\psi \in \Omega^k$. The subspaces $\varepsilon = 1$ and $\varepsilon = -1$ have equal dimension 2^{n-1} .

6.2. Discretization On the de Rham Complex

If we want to consider the approximation of some continuous object on a general mesh, the exterior bundle is a very natural object. Indeed, k -forms may be integrated over k -dimensional surfaces. Thus, if a general mesh (a cell complex) is given, a k -form defines a function on the k -dimensional cells of the mesh in a very natural way. For each cell we can define its "characteristic form" on the mesh as the equivalence class of forms with $\chi^i(c_j) = \delta_j^i$ and decompose the general form on the mesh as $f(c) = f_i \chi^i(c)$. The external derivative defines in a similar natural way a derivative for functions on the mesh, with the same most important exact property $d^2 = 0$.

For the definition of the $*$ -operator we need a metric and a dual mesh. The metric defines in a natural way for every cell c_i its area $a_i = a(c_i) > 0$. In the Euclidean case and a triangulation, these values depend on each other, but in the general case they may be considered as independent variables which define the metric. In the following we consider them as given and defining the metric.

The dual mesh is a quite natural mesh which can be defined for every mesh: for each k -dimensional cell c_i of the original mesh it has a $(n-k)$ -dimensional dual cell \hat{c}_i which intersects the original cell in a single point orthogonally and with positive intersection index. Now, the metric defines as well a scalar value \hat{a}_i for each cell of the dual mesh \hat{c}_i — the area of the dual cell. Now, the $*$ -operator on the mesh may be defined as

$$(*f)(c) = \frac{\hat{a}_i}{a_i} f_i \chi^i(c). \quad (28)$$

Note that the dual of the dual mesh has the same cells, but in some dimensions with different orientations. Therefore, for $*^2$ there appears the factor $(-1)^{k(n-k)}$ as in the continuous case.

Note also that the discrete exterior bundle is connected in a simple way with a natural "regular refinement" step of a mesh: To refine a mesh in a regular way, we can put one node in the center of each cell, and connect nodes of neighbour dimension cells with edges if one of the cell is on the boundary of the other. This leads to a mesh where each cell has the topological structure of a unit hypercube.

For a simple hypercubic lattice on a flat background, this discretization scheme reduces to our proposal above.

7. Conclusion

The approach described here may help to solve some outstanding problems of chiral gauge theory.

First, there is the general problem how to formulate a chiral gauge theory on the lattice, known as the "regularization problem": any consistent regularization that preserves the gauge symmetry must refer to the fermion representation [12]. In our case, we describe chiral gauge theory only in a very special, two-dimensional representation, therefore it is not in conflict with this requirement. And it gives a representation we need in the standard model to describe the $SU(2)$ part of the electroweak interaction. Moreover, if neutrinos are Dirac particles, we don't need any other representations – only their combinations (one lepton and three quark doublets). Of course, a lot of things have to be done to obtain a complete quantum chiral gauge theory in this way. The problem how to handle the $U(1)$ part of electroweak interaction, which needs the incorporation of all fermion doublets, is completely open.

Second, it may be helpful for a solution of the fermion doubling problem (see [6] for an introduction, [7,9,12,17] for various approaches to solve this problem in chiral gauge theory). It reduces the number of "doubled flavours" from four in the staggered fermion approach (which seem unphysical) to two which already allow a meaningful physical interpretation as lepton or quark doublets. The nontrivial character of the complex structure may prevent the application of standard no-go-theorems like the famous Nielsen-Ninomiya [10] theorem.

We also obtain an interesting representation of fermions on a general metric background which does not need tetrad or triad variables for the gravitational field. Instead, the gravitational field may be described as usual by the metric tensor $g_{\mu\nu}$.

Acknowledgements

Thanks to M. Nobes, L. Bourhis and S. Carlip for interesting discussions.

Appendix A. Connection to the General Lorentz Ether

The approach to fermions presented here has been developed to solve a particular problem connected with the description of fermion fields General Lorentz Ether Theory (GLET), which we want to describe here shortly.

This theory has been proposed by the author in [15] as a generalization of the classical Lorentz ether to gravity. It is a metric theory of gravity, where the "effective" metric $g_{\mu\nu}$ is defined by a variant of the ADM decomposition from classical condensed matter (ether) variables: density $\rho = g^{00}\sqrt{-g} > 0$, velocity $v^i = g^{0i}/g^{00}$, and a three-dimensional pressure tensor $p^{ij}\sqrt{p} = -g^{ij}\sqrt{g}$. The continuity and Euler equations

of classical continuum mechanics

$$\partial_t \rho + \partial_i(\rho v^i) = 0, \tag{A.1}$$

$$\partial_t(\rho v^j) + \partial_i(\rho v^i v^j + p^{ij}) = 0 \tag{A.2}$$

transforms into the harmonic condition $\partial_\mu g^{\mu\nu} \sqrt{-g} = 0$ for the effective metric. The Lagrangian of GLET consists of the classical GR Lagrangian and some non-covariant term which describes the dependence on the Newtonian background:

$$L = L_{GR} + L_{matter}(g_{\mu\nu}, \varphi^m) - (8\pi G)^{-1}(\Upsilon g^{00} - \Xi \delta_{ij} g^{ij})\sqrt{-g}. \tag{A.3}$$

Therefore, the theory has a well-defined GR limit $\Upsilon, \Xi \rightarrow 0$ which makes the theory viable. This Lagrangian has been derived from first principles which seem quite natural for condensed matter theories in [15]. Especially the covariant character of its matter part, that means the Einstein equivalence principle, follows.

It has been noted [2] that this derivation does not extend (at least not in any obvious way) to tetrad or triad variables. Because the standard way to describe fermions on a curved background uses tetrad (or, in the context of an ADM decomposition, triad) variables, this makes the incorporation of fermions into this theory problematic. One way to solve this problem is to find a way to describe fermions without tetrad or triad variables, what has been done in our approach: Here we define a pair of Dirac fermions as classical tensor fields.

It should also be noted that the regularization problem of chiral gauge theory as well as the fermion doubling problem become important in ether theory if we want to use an "atomic ether" concept to regularize the theory. An "atomic ether" regularization, which interprets the density ρ as a particle density and velocity v^i as their average velocity is a quite natural way to quantize an ether theory. But in this case we need some lattice discretization for all parts of the standard model [13].

In the context of the ether approach, an extremely simple toy model would be a cubic ether crystal. Distortions of this crystal would be defined by three real values per node. This toy model exactly fits into the lattice discretization described here, which gives three generations of a pair of fermions. Thus, already an extremely simple toy model seems to be able to explain the three fermion generations. Moreover it suggests that fermions appear in pairs, with a natural possibility for $1 + \gamma^5$ type gauge interactions inside these pairs. These are already nontrivial and important features of the standard model. Especially if neutrinos appear to be Dirac particles, this scheme holds for all fermions of the standard model.

Appendix B. Compatibility

with ADM Decomposition

For the compatibility of the approach described here with GLET it is the compatibility with the ADM decomposition is crucial and worth to be considered. Using "comoving" spatial coordinates (which remain constant along the velocity field v^i), the harmonic operator of the metric $g_{\mu\nu}$ reduces to

$$\square\psi = -(\rho\partial_t^2 - \Delta)\psi \quad (\text{B.1})$$

where Δ is the standard three-dimensional (harmonic) Laplace operator of the metric p_{ij} . Thus, we ADM decomposition allows to reduce the harmonic equation to a "three-and-one-half-dimensional" equation

$$\sqrt{\rho}\partial_t\psi = \sqrt{\Delta}\psi. \quad (\text{B.2})$$

Now it seems natural to use a mesh which is compatible with the ADM decomposition. We start with an arbitrary mesh on the spatial slice at a given time t_0 . On this three-dimensional mesh we can define the Dirac operator following the scheme described above. Then we can extend this mesh into a four-dimensional mesh. First, we extend the mesh nodes at t_0 following the comoving spatial coordinates into comoving world-lines. For the preferred time we use a global, regular lattice and simple central differences. Such a choice of a lattice and discretization is in natural agreement with a crystalline ether model.

Appendix C. A Natural $SO(3)$ Gauge Field for a General Three-Dimensional Mesh

Another reason why we think that the focus of three spatial dimensions will be helpful is that a general three-dimensional mesh defines in a simple and natural way an $SO(3)$ gauge field. Because at the Lie algebra level $so(3) = su(2)$ this may be a candidate for the $su(2)$ gauge field of the electroweak interaction.

To understand this gauge connection, remember that to define a gauge connection it is sufficient to define its integral over closed loops. Thus, let's define the integral over a closed loop starting from a general point (that means, inside a cell) and of general type (thus, the loop intersects only planar cells of the mesh).

Now, assume also that we have made a simple regular refinement step, therefore, the mesh is a cubic mesh. For each cubic cell, we fix some standard map into the regular cube. Then we start with a standard reper (as it comes from the standard cube) and use trivial parallel transport of the reper (as defined on the standard cube) inside the cells up to the boundary. On the boundary, one vector of the reper is always orthogonal to the boundary. If we come back to the original node, the transferred reper is also, by

construction, a reper in this cube, its directions are parallel to the original reper. Nonetheless, it is in general a rotated reper. This rotation defines the gauge connection. It measures the irregularity of the mesh: For a regular cubic mesh it is trivial.

Now, the existence of a gauge field which is in such a natural way connected with mesh irregularity does not look like an accident of nature. Our hypothesis is that it is (at least connected in some way with) the $SU(2)$ gauge field of the electroweak interaction. The details of such a connection are subject of future research.

References

- [1] N.D. Birrell, P.C.W. Davies, Quantum fields in curved space, Cambridge University Press, Cambridge (1982)
- [2] L. Bourhis, personal communication
- [3] T.-W. Chiu, Phys.Lett. B 521, 429 (2001)
- [4] K. Fujikawa, Nucl.Phys.B 589, 487 (2000)
- [5] P.H.Ginsparg, K.G.Wilson, Phys.Rev.D 25, 2649 (1982)
- [6] R. Gupta, Introduction to lattice QCD, hep-lat/9807028
- [7] W. Kerler, More chiral operators on the lattice, hep-lat/0204008
- [8] J. Kogut, L. Susskind, Phys. Rev. D11, 395 (1975)
- [9] R. Narayanan, Chiral gauge theories in the overlap formalism, hep-lat/9707035
- [10] H. Nielsen, M. Ninomiya, Nucl.Phys.B 183, 20 (1981)
- [11] H. Neuberger, Phys.Lett. B 417, 141, (1998)
- [12] M. Lüscher, hep-th/0102028
- [13] M. Nobes, personal communication
- [14] G. Pete, Morse theory, <http://www.math.u-szeged.hu/gpete/morse.ps>
- [15] I. Schmelzer, gr-qc/0205035
- [16] Y. Shamir, Nucl. Phys. B 406, 90 (1993)
- [17] M. Testa, The Rome approach to chirality, hep-lat/9707007