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Classical Electrodynamical Aspect of the Dirac Equation

Contents

Abstract

It is shown that the Dirac equation may describe a coupled system of electromagnetic (in the terms of field strengths) and scalar fields. The unitary relationship between the slightly generalized Maxwell equations with gradient-type 4-current and the Dirac equation is found. It is demonstrated that together with spinor symmetry the massless Dirac equation has also Poincare tensor-scalar and vector symmetries. In the considered Maxwell equations an interaction with external potential field, modeling some medium, is introduced. It is shown that the solutions of the Maxwell equations in such medium reconstruct the discrete states and the relativistic spectra of the hydrogen-like atoms. These facts are interpreted as the consequences of the unitary relationship between the Dirac and slightly generalized Maxwell equations. Thus, the electrodynamical aspect of the Dirac equation is demonstrated.

1. Introduction

On the basis of unitary relationship between the Dirac and the slightly generalized Maxwell equations we are able to show the new features of the well known Dirac equation. We demonstrate the electrodynamical aspect of the Dirac equation. It is shown that this equation may describe the coupled system of electromagnetic (in the terms of field strengths) and scalar fields.

The first step in our consideration is the unitary relationship (and wide range analogy) between the Dirac equation and slightly generalized Maxwell equations [1, 2].

Our second step is the symmetry principle. On the basis of this principle we introduced in [3, 4] the most symmetrical form of generalized Maxwell equations which now can describe both bosons and fermions because they have (see [3, 4]) both spin 1 and spin $1/2$ symmetries. On the other hand, namely these equations are unitarily connected with the Dirac equation.

In our third step we refer to Sallhofer, who suggested in [5–7] the possibility of introduction of interaction with external field as the interaction with specific media in the Maxwell theory (a new way of introduction of the interaction into the field equations). Nevertheless, our model of atom (and of electron) [1, 2] is essentially different from the Sallhofer's one.

On the basis of these three main ideas we are able to demonstrate the electrodynamical aspect of the Dirac theory of atom and of atomic electron.

The interest to the problem of relationship between the Dirac and Maxwell equations dates back to the time of creation of quantum mechanics [8–18]. But the authors of these papers considered the most simple example of free and massless Dirac equation. The interest to this relationship has grown in recent years due to the results [5–7], where the investigations of the case $\mathbf{m}_0 \neq 0$ and the interaction potential $\Phi \neq 0$ were started. Another approach was developed in [19–26], where the quadratic relations between the fermionic and bosonic amplitudes were found and used. In our above mentioned papers [1–4], in publications [27–32] and herein we consider the linear relations between the fermionic and bosonic amplitudes. In [27–32] we have found the relationship between the symmetry properties of the Dirac and Maxwell equations, the complete set of 8 transformations linking these equations, the relationship between the conservation laws for the electromagnetic and spinor fields, the relationship between the Lagrangians for these fields. Here we summarize our previous results and give some new details (Lamb shift consideration) of the inneratomic electrodynamics and its application to the hydrogen atom. The facts under consideration we interpret below on the basis of the Dirac (not Maxwell) equation.

2. Slightly Generalized Maxwell Equations with Maximal Symmetry Properties and Their Relationship with the Massless Dirac Equation

Consider the Maxwell equations with specific (gradient-type) form of electric and magnetic sources and the symmetry properties of such equations. Namely such equations are shown [1–4] to be directly related to the massless Dirac equation. Corresponding equations for the system of electromagnetic and scalar fields ($\vec{E}, \vec{H}, E^0, H^0$) have the form:

$$\begin{aligned} &\partial_0 \vec{E} = \mathrm{curl}\, \vec{H} - \mathrm{grad}\, E^0, \\ &\partial_0 \vec{H} = -\mathrm{curl}\, \vec{E} - \mathrm{grad}\, H^0, \\ &\mathrm{div} \vec{E} = -\partial_0 E^0, \\ &\mathrm{div} \vec{H} = -\partial_0 H^0. \end{aligned} \tag{1}$$

The Eqs. (1) are nothing more than the slightly generalized Maxwell equations with gradient-like electric and magnetic sources $j^e_\mu = -\partial_\mu E^0$, $j^{mag}_\mu = -\partial_\mu H^0$, i.e.

$$\begin{aligned} &\vec{j}_e = \mathrm{grad}\, E^0, \qquad \vec{j}_{mag} = \mathrm{grad}\, H^0, \\ &\rho_e = -\partial_0 E^0, \qquad \rho_{mag} = -\partial_0 H^0. \end{aligned} \tag{2}$$

In terms of complex 4-component object $\mathcal{E} \equiv (\mathcal{E}^\mu) = E - \mathrm{i}H$

$$\mathcal{E} \equiv \left| \begin{array}{c} \vec{\mathcal{E}} \\ \mathcal{E}^0 \end{array} \right| = \mathrm{column}\left|E^1 - \mathrm{i}H^1, E^2 - \mathrm{i}H^2, E^3 - \mathrm{i}H^3, E^0 - \mathrm{i}H^0\right| \tag{3}$$

where $\vec{\mathcal{E}} = \vec{E} - \mathrm{i}\vec{H}$ is the well-known complex form for the electromagnetic field used by Majorana as far back as near 1930 (see, e.g., [9]), Eqs. (1) may be rewritten in the manifestly covariant form

$$\partial_\mu \mathcal{E}_\nu - \partial_\nu \mathcal{E}_\mu + \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\partial^\rho \mathcal{E}^\sigma = 0, \quad \partial_\mu \mathcal{E}^\mu = 0. \tag{4}$$

It is useful also to consider the following form of Eqs. (1), (4):

$$(\mathrm{i}\partial_0 - \vec{S}\cdot\vec{p})\vec{\mathcal{E}} - \mathrm{i}\mathrm{grad}\, \mathcal{E}^0 = 0, \quad \partial_\mu \mathcal{E}^\mu = 0, \tag{5}$$

where $\vec{S} \equiv (S^j)$ are the generators of irreducible representation $D(1)$ of the group $SU(2)$, i.e. the quantum-mechanical spin 1 operator:

$$\begin{aligned} &S^1 = \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{array} \right|, \\ &S^2 = \left| \begin{array}{ccc} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{array} \right|, \\ &S^3 = \left| \begin{array}{ccc} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right|, \\ &\vec{S}^2 = 1(1+1)I. \end{aligned} \tag{6}$$

Equations (1),(4),(5) are directly connected with the free massless Dirac equation

$$\mathrm{i}\gamma^\mu \partial_\mu \Psi(x) = 0. \tag{7}$$

The substitution of

$$\begin{aligned} &\psi = \left| \begin{array}{c} E^3 + \mathrm{i}H^0 \\ E^1 + \mathrm{i}E^2 \\ \mathrm{i}H^3 + E^0 \\ -H^2 + \mathrm{i}H^1 \end{array} \right| = U\mathcal{E}, \\ &U = \left| \begin{array}{cccc} 0 & 0 & C_+ & C_- \\ C_+ & \mathrm{i}C_+ & 0 & 0 \\ 0 & 0 & C_- & C_+ \\ C_- & \mathrm{i}C_- & 0 & 0 \end{array} \right|, \\ &C_\mp \equiv \frac{1}{2}(C \mp 1), \quad C\Psi \equiv \Psi^*, \quad C\mathcal{E} \equiv \mathcal{E}^* \end{aligned} \tag{8}$$

into Dirac equation (7) with γ matrices in standard Pauli-Dirac representation guarantees its transformation into the generalized Maxwell equations (1),(4),(5). Thus, the equations (1),(4),(5) may be interpreted as the electrodynamical representation of the massless Dirac equation.The complete set of 8 transformations like (8), which relate generalized Maxwell equations (4) and massless Dirac equation (7), was found in [27,28]. Unitary relationship between the generalized Maxwell equations (4) and massless Dirac equation (7) can be found in our papers [30–32].

The unitarity of the operator (8) can be verified easily by taking into account that the equations

$$(AC)^{\dagger} = CA^{\dagger}, \quad aC = Ca^{*}, \quad (aC)^{*} = Ca \tag{9}$$

hold for an arbitrary matrix A and a complex number a. We note that in the real algebra (i. e. the algebra over the field of real numbers) and in the Hilbert space of quantum mechanical amplitudes this operator has all properties of unitarity.

Equations (4) (or their another representations (1),(5),(7)) are the maximally symmetrical form of the generalized Maxwell equations. We consider here representation (4) as an example. On the other hand it is the electrodynamical representation of the massless Dirac equation. The following theorem is valid.

Theorem. *The generalized Maxwell equations (4) (the massless Dirac equation (7)) are (is) invariant with respect to the three different transformations, which are generated by three different representations* P^V, P^{TS}, P^S *of the Poincaré group* $P(1,3)$ *given by the formulae*

$$\begin{aligned} &\mathcal{E}(x) \to \mathcal{E}^V(x) = \Lambda\mathcal{E}(\Lambda^{-1}(x-a)),\\ &\mathcal{E}(x) \to \mathcal{E}^{TS}(x) = F(\Lambda)\mathcal{E}(\Lambda^{-1}(x-a)),\\ &\mathcal{E}(x) \to \mathcal{E}^S(x) = S(\Lambda)\mathcal{E}(\Lambda^{-1}(x-a)) \end{aligned} \tag{10}$$

where Λ *is a vector (i. e.* $(1/2,1/2)$*),* $F(\Lambda)$ *is a tensor-scalar* $((0,1)\otimes(0,0))$ *and* $S(\Lambda)$ *is a spinor representation (*$(0,1/2)\otimes(1/2,0)$*) of* $SL(2,C)$ *group. This means that the equations (4) (or (7)) have both spin 1 and spin 1/2 symmetries.*

Proof. Let us write the infinitesimal transformations, following from (10), in the form

$$\mathcal{E}^{V,TS,S}(x) = (1 - a^{\rho}\partial_{\rho} - \frac{1}{2}\omega^{\rho\sigma}j^{V,TS,S}_{\rho\sigma})\mathcal{E}(x). \tag{11}$$

Then the generators of the transformations (11) have the form

$$\partial_{\rho} = \frac{\partial}{\partial x^{\rho}}, \quad j^{V,TS,S}_{\rho\sigma} = x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho} + s^{V,TS,S}_{\rho\sigma} \tag{12}$$

where

$$(s^V_{\rho\sigma})^{\mu}_{\nu} = \delta^{\mu}_{\rho}g_{\sigma\nu} - \delta^{\mu}_{\sigma}g_{\rho\nu}, \quad s^V_{\rho\sigma} \in \left(\frac{1}{2},\frac{1}{2}\right),$$

$$\begin{aligned} &s^{TS}_{\rho\sigma} = \begin{vmatrix} s^T_{\rho\sigma} & 0\\ 0 & 0 \end{vmatrix} \in (1,0)\oplus(0,0),\\ &\quad s^T_{\rho\sigma} = -s^T_{\sigma\rho} : \, s^T_{mn} = -\mathrm{i}\varepsilon^{mnj}S^j, \, s^T_{0j} = S^j, \end{aligned} \tag{13}$$

(S^j are given by the formula (6)) and

$$s^S_{\rho\sigma} = \frac{1}{4}[\widetilde{\gamma}_{\rho},\widetilde{\gamma}_{\sigma}], \quad \widetilde{\gamma} = U^{\dagger}\gamma U, \tag{14}$$

(the unitary operator U is given by the formula (8), $\widetilde{\gamma}$ matrices here may be easily found according to (14)). The explicit form of $\widetilde{\gamma}$ matrices is following:

$$\begin{aligned} \widetilde{\gamma}^0 &= \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{vmatrix} C,\\ \widetilde{\gamma}^1 &= \begin{vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -\mathrm{i} & 0\\ 0 & \mathrm{i} & 0 & 0\\ -1 & 0 & 0 & 0 \end{vmatrix} C,\\ \widetilde{\gamma}^2 &= \begin{vmatrix} 0 & 0 & \mathrm{i} & 0\\ 0 & 0 & 0 & 1\\ -\mathrm{i} & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{vmatrix} C,\\ \widetilde{\gamma}^3 &= \begin{vmatrix} 0 & -\mathrm{i} & 0 & 0\\ \mathrm{i} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{vmatrix} C. \end{aligned} \tag{15}$$

We call the representation (15) the bosonic representation of the γ matrices. In this representation the imaginary unit i is represented by the 4×4 matrix operator.

$$\begin{aligned} &V\mathrm{i}V^{\dagger} = \mathrm{i}\Gamma,\\ &\Gamma \equiv \begin{vmatrix} 0 & \mathrm{i} & 0 & 0\\ -\mathrm{i} & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0 \end{vmatrix} = \Gamma^{\dagger} = \Gamma^{-1},\\ &\Gamma^2 = 1. \end{aligned} \tag{16}$$

Due to the unitarity of the operator U in (8), the $\widetilde{\gamma}^{\mu}$ matrices not only obey the Clifford-Dirac algebra

$$\widetilde{\gamma}^{\mu}\widetilde{\gamma}^{\nu} + \widetilde{\gamma}^{\nu}\widetilde{\gamma}^{\mu} = 2g^{\mu\nu} \tag{17}$$

but also have the same Hermitian properties as the Pauli-Dirac γ^{μ} matrices:

$$\widetilde{\gamma}^{0\dagger} = \widetilde{\gamma}^0, \quad \widetilde{\gamma}^{k\dagger} = -\widetilde{\gamma}^k. \tag{18}$$

Thus, the formulae (15) give indeed an exotic representation of γ^{μ} matrices.

Now the proof of the theorem is reduced to the verification that all the generators (12) obey the commutation relations of the $P(1,3)$ group and commute with the operator of the generalized Maxwell equations (4)=(5), which can be rewritten in the Dirac form

$$\widetilde{\gamma}^{\mu}\partial_{\mu}\mathcal{E}(x) = 0 \tag{19}$$

(for some details see Refs. [3,4]). **QED.**

This result about the generalized Maxwell equations (4) (the Dirac equation (7) in electrodynamical representation) means the following. From group theoretical point of view these equations can describe both bosons and fermions. This means that one has direct group-theoretical grounds to apply these equations for the description of electron, as it is presented below in Sec. 5. However, for the Dirac equation this situation is ordinary.

A distinctive feature of the equation (4) for the system $\mathcal{E} = (\vec{\mathcal{E}}, \mathcal{E}^0)$ (i.e. for the system of interacting irreducible $(0,1)$ and $(0,0)$ fields) is the following. It is the manifestly covariant equation with minimal number of components, i. e. the equation without redundant components for this system.

Note that each of the three representations (10) of the $P(1,3)$ group is a local one, because each matrix part of transformations (10) (matrices Λ, $F(\Lambda)$ and $S(\Lambda)$) does not depend on coordinates $x \in R^4$, and, consequently, the generators (12) belong to the Lie class of operators. Each of the transformations in (10) may be understood as connected with special relativity transformations in the space-time $R^4 = \{x\}$, i.e. with transformations in the manifold of inertial frame of references.

We emphasize that the equation (19) has the form of massless Dirac equation for fermions field. In such consideration of equation (19) the $\widetilde{\gamma}^\mu$ matrices may be chosen in arbitrary representation (e.g., in each of representations of Pauli-Dirac, Majorana, Weyl, ...). However, only in exotic representation (15) equation (19) is the Maxwell equation for the system of interacting electromagnetic $\vec{\mathcal{E}} = \vec{E} - \mathrm{i}\vec{H}$ and scalar $\mathcal{E}^0 = E^0 - \mathrm{i}H^0$ fields (therefore, we have called the representation (15) the bosonic one). Thus, if one considers the equation (19) as bosonic one, the representation of γ^μ matrices and their explicit form *must be fixed* in the form (15). In the case of bosonic interpretation of Eq. (7) one must fixes the explicit form of γ^μ in standard Pauli-Dirac representation and fixes the form of Ψ as (8).

It follows from the Eqs. (4) that the field $\mathcal{E} = (\vec{\mathcal{E}}, \mathcal{E}^0)$ is massless, i.e. $\partial^\nu \partial_\nu \mathcal{E}^\mu = 0$. Therefore it is interesting to note that neither P^V, nor P^{TS} symmetries cannot be extended to the local conformal $C(1,3)$ symmetry. Only the spinor C^S representation of $C(1,3)$ group, obtained from the local P^S representation, is the symmetry group for the generalized Maxwell equations (4). This fact is understandable: the electromagnetic field $\vec{\mathcal{E}} = \vec{E} - \mathrm{i}\vec{H}$ obeying Eqs. (4) is not free, it interacts with the scalar field $\mathcal{E}^0$.

Consider the particular case of standard (non-generalized) Maxwell equations, namely, the case of equations (4) without magnetic charge and current densities, i. e. the case when $H^0 = 0$ but $E^0 \neq 0$. The symmetry properties of such standard equations are strongly restricted in comparison with the generalized Eqs. (4): they are invariant only with respect to tensor-scalar (spins 1 and 0) representation of Poincaré group defined by the corresponding representation $(0,1) \otimes (0,0)$ of proper ortochronous Lorentz group $SL(2,C)$. Another symmetries mentioned in the theorem are lost for this case too. The proof of this assertion follows from the fact that the vector $((1/2,1/2))$ and the spinor $((0,1/2) \oplus (1/2,0))$ transformations of $\mathcal{E} = (\vec{\mathcal{E}}, \mathcal{E}^0)$ mix the $\mathcal{E}^0$ and $\vec{\mathcal{E}}$ components of the field $\mathcal{E}$, and only the tensor-scalar $(0,1) \oplus (0,0)$ transformations do not mix them.

For the free Maxwell equation in vacuum without sources (the case $E^0 = H^0 = 0$) the losing of above mentioned symmetries is evident from the same reasons. Moreover, it is well known that such equations are invariant only with respect to tensor (spin 1) representations of Poincaré and conformal groups and with respect to dual transformation: $\vec{E} \rightarrow \vec{H}, \vec{H} \rightarrow -\vec{E}$. We have obtained the extended 32-dimensional Lie algebra [33] (and the corresponding group) of invariance of free Maxwell equations, which is isomorphic to $C(1,3) \oplus C(1,3) \oplus dual$ algebra. We were successful to prove it appealing not to Lie class of symmetry operators but to a more general, namely, to the simplest Lie-Backlund class of operators. The corresponding generalization of symmetries of Eqs. (4) presented in the above theorem leads to a wide 246-dimensional Lie algebra in the class of first order Lie-Backlund operators. Thus, the Maxwell equations (4) with electric and magnetic gradient-like sources have the maximally possible symmetry properties among the standard and generalized equations of classical electrodynamics.

The general solution of Eqs. (4) was found in [31,32] directly by the application of Fourier method. In terms of helicity amplitudes $c^\mu(\vec{k})$ this solution has the form

$$\mathcal{E}(x) = \int \mathrm{d}^3k \sqrt{\frac{2\omega}{(2\pi)^3}} \left\{ \left[c^1 e_1 + c^3 (e_3 + e_4) \right] \mathrm{e}^{-ikx} + \left[c^{*2} e_1 + c^{*4} (e_3 + e_4) \right] \mathrm{e}^{ikx} \right\} \quad (20)$$

where 4-component basis vectors e_α are taken in the form

$$e_1 = \begin{vmatrix} \vec{e_1} \\ 0 \end{vmatrix}, \quad e_2 = \begin{vmatrix} \vec{e_2} \\ 0 \end{vmatrix}, \quad e_3 = \begin{vmatrix} \vec{e_3} \\ 0 \end{vmatrix}, \quad e_4 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}. \quad (21)$$

Here the 3-component basis vectors which, without any loss of generality, can be taken as

$$\vec{e_1} = \frac{1}{\omega\sqrt{2(k^1k^1 + k^2k^2)}} \begin{vmatrix} \omega k^2 - \mathrm{i}k^1k^3 \\ -\omega k^1 - \mathrm{i}k^2k^3 \\ \mathrm{i}(k^1k^1 + k^2k^2) \end{vmatrix}, \quad \vec{e_2} = \vec{e_1}^{\,*}, \quad \vec{e_3} = \frac{\vec{k}}{\omega}, \quad (22)$$

are the eigenvectors for the quantum-mechanical helicity operator for the spin $s = 1$.

Remark. The general first-order differential equation in partial derivatives for the coupled system of tensor fields, which contain both massless and non-zero mass Dirac equations, was considered by Madelung and Lanczos [34–36] (both in quaternion or ordinary treatment). Recently this equation was considered in [37, 38]. In the paper [38] the solutions of Madelung-Lanczos (Dirac-Kahler) equation in the form of matrix-dyads were found and some additional symmetries of this equation were considered. Such equation is the system of 16 equations for 16 (real) functions. Namely, it is the coupled system of field strengths $(\vec{E}, \vec{H})$, scalar fields (E^0, H^0) and potentials (together with dual potentials) $A = (A^\mu)$, $\widetilde{A} = (\widetilde{A}^\mu)$.

For the massless case this equation was called as the generalized Maxwell equations (see [37] and references there in). Such generalized Maxwell equations from [37] are the system of 11 equations for 11 (real) functions. Namely, it is the coupled system of field strengths $(\vec{E}, \vec{H})$, one scalar field (E^0) and potentials $A = (A^\mu)$. It is the subsystem of Madelung - Lanczos (Dirac - Kahler) equation with only one real scalar field and only ordinary potentials.

We have considered in [1–4, 31] (and here) the subsystem of above mentioned system for another object. We deal with the system of 8 equations for 8 (real) functions, or with 4 equations for 4 complex functions. Such equations contain only field strengths $(\vec{E}, \vec{H}, E^0, H^0)$ of electromagnetic-scalar field and not contain the potentials $A = (A^\mu)$, $\widetilde{A} = (\widetilde{A}^\mu)$. In this sense considered by us system of equations is the minimal generalization of the ordinary Maxwell equations (hence, we called it *slightly generalized Maxwell equations* (SGME) [39]). The subsystem of our SGME system with only one real scalar field $E^0 \neq 0$, $H^0 = 0$, (its solutions and symmetries) was also considered in our papers [1–4,31] , especially in [32,40].

One of the distinctive features of our (SGME) system of equations is following. Such system is invariant with respect to the three essentially different local representations of one and the same fundamental relativistic group, i. e. of the Poincaré group (see the corresponding theorem above and also [3, 4]). The most interesting distinctive feature is that namely these equations (SGME) allowed us to describe (after introducing into them the interaction with external field in the form of Sallhofer's permeabilities [5–7]) the quantum stationary states of the hydrogen atom (see the Sec.3 below). (End of remark).

Note that if the quantities E^0, H^0 in Eqs. (1) are some given functions for which the representation

$$E^0 - \mathrm{i}H^0 = \int \mathrm{d}^3k \sqrt{\frac{2\omega}{(2\pi)^3}} \left(c^3 \mathrm{e}^{-\mathrm{i}kx} + c^4 \mathrm{e}^{\mathrm{i}kx}\right) \quad (23)$$

is valid, then Eqs. (1) are the Maxwell equations with the given sources, $j^e_\mu = -\partial_\mu E^0, j^{mag}_\mu = -\partial_\mu H^0$ (namely these 4 currents we call the gradient-like sources). In this case the general solution of the Maxwell equations (1)=(4) with the given sources, as follows from (20), has the form

$$\begin{aligned} \vec{E}(x) &= \int \mathrm{d}^3k \sqrt{\frac{\omega}{2(2\pi)^3}} \left(c^1 \vec{e}_1 \right. \\ &\qquad \left. + c^2 \vec{e}_2 + \alpha \vec{e}_3\right) \mathrm{e}^{-\mathrm{i}kx} + c.c \\ \vec{H}(x) &= \mathrm{i} \int \mathrm{d}^3k \sqrt{\frac{\omega}{2(2\pi)^3}} \left(c^1 \vec{e}_1 \right. \\ &\qquad \left. - c^2 \vec{e}_2 + \beta \vec{e}_3\right) \mathrm{e}^{-\mathrm{i}kx} + c.c \end{aligned} \quad (24)$$

where the amplitudes of longitudinal waves $\vec{e}_3 \exp(-\mathrm{i}kx)$ are $\alpha = c^3 + c^4$, $\beta = c^3 - c^4$ and c^3, c^4 are determined by the functions E^0, H^0 according to the Eq. (23).

The longitudinal electromagnetic waves were the object of long time investigation of Hvorostenko [41]. Here we are able (i) to add to his results the exact solution of the Maxwell equations with gradient-like sources which contains such a waves and (ii) to make the assertion about location of these waves in the same space-time area where the gradient-like sources are located (the reason: the amplitudes c^3, c^4 which define this waves and the gradient-like sources are the same).

Note that on arbitrary step of procedure of finding the solution (20) one may put $H^0 = 0$, or $c^4 = 0$, and consider in such easy way the partial case with only one scalar field $E^0 \neq 0$.

Now, knowing the operator U (8), it is easy to obtain the relationship between the amplitudes $a^r(\vec{k})$, $b^r(\vec{k})$ determining the well known fermionic solution of the massless Dirac equation, and the amplitudes $c^\alpha(\vec{k})$, determining the bosonic solution (20). Corresponding formulae related fermionic and bosonic amplitudes were found in [31, 32]. Therefore, the fermionic states may be constructed over bosonic states. This assertion finishes the consideration of group-theoretical grounds of our model.

3. New Classical Electrodynamical Hydrogen Atom Model

The generalized Maxwell equations (1) may be extended on the case of specific inneratomic medium. Namely these equations are put into the ground of our model of atom.

Consider the slightly generalized Maxwell equations with gradient-type sources in a medium:

$$\begin{aligned} &\operatorname{curl}\vec{H} - \partial_0 \varepsilon \vec{E} = \vec{j}_e, \\ &\operatorname{curl}\vec{E} + \partial_0 \mu \vec{H} = \vec{j}_{mag}, \\ &\operatorname{div}\varepsilon \vec{E} = \rho_e, \\ &\operatorname{div}\mu \vec{H} = \rho_{mag}, \end{aligned} \tag{25}$$

where $\vec{E}$ and $\vec{H}$ are the electromagnetic field strengths, ε and μ are the electric and magnetic permeabilities of the medium being the same as in the electrodynamical hydrogen atom model of H. Sallhofer [5–7, 42]:

$$\begin{aligned} &\varepsilon\,(\vec{x}) = 1 - \frac{\Phi\,(\vec{x}) + \mathbf{m}_0}{\omega}, \\ &\mu\,(\vec{x}) = 1 - \frac{\Phi\,(\vec{x}) - \mathbf{m}_0}{\omega} \end{aligned} \tag{26}$$

where $\Phi\,(\vec{x}) = -Ze^2/r$ (we use the units: $\hbar = c = 1$, transition to standard system is fulfilled by the substitution $\omega \longrightarrow \hbar\omega$, $\mathbf{m}_0 \longrightarrow \mathbf{m}_0 c^2$)). The current and charge densities in equations (25) have the form

$$\begin{aligned} &\vec{j}_e = \operatorname{grad} E^0, \\ &\vec{j}_{mag} = -\operatorname{grad} H^0, \\ &\rho_e = -\varepsilon\mu\partial_0 E^0 + \vec{E}\operatorname{grad}\varepsilon, \\ &\rho_{mag} = -\varepsilon\mu\partial_0 H^0 + \vec{H}\operatorname{grad}\mu \end{aligned} \tag{27}$$

where E^0, H^0 is the pair of functions (two real scalar fields) generating the densities of gradient-like sources.

One can easily see that equations (25) are not ordinary electrodynamical equations known from the Maxwell theory. These equations have the additional terms which can be considered as the magnetic current and charge densities — in one possible interpretation, or equations (25) can be considered as the equations for compound system of electromagnetic ($\vec{E}$, $\vec{H}$) and scalar E^0, H^0 fields in another possible interpretation.

Contrary to [1, 2], here the equations (25) are solved directly by means of separation of variables method. It is useful to rewrite these equations in the mathematically equivalent form where the sources are maximally simple:

$$\begin{aligned} &\operatorname{curl}\vec{H} - \varepsilon\partial_0 \vec{E} = \vec{j}_e, \\ &\operatorname{curl}\vec{E} + \mu\partial_0 \vec{H} = \vec{j}_{mag}, \\ &\operatorname{div}\vec{E} = \tilde{\rho}_e, \\ &\operatorname{div}\vec{H} = \tilde{\rho}_{mag} \end{aligned} \tag{28}$$

where

$$\begin{aligned} &\vec{j}_e = \operatorname{grad} E^0, \quad \vec{j}_{mag} = -\operatorname{grad} H^0, \\ &\tilde{\rho}_e = -\mu\partial_0 E^0, \quad \tilde{\rho}_{mag} = -\varepsilon\partial_0 H^0. \end{aligned} \tag{29}$$

Consider the stationary solutions of equations (28). Assuming the harmonic time dependence for the functions E^0, H^0

$$\begin{aligned} E^0(t,\vec{x}) &= E^0_A(\vec{x})\cos\omega t + E^0_B(\vec{x})\sin\omega t, \\ H^0(t,\vec{x}) &= H^0_A(\vec{x})\cos\omega t + H^0_B(\vec{x})\sin\omega t, \end{aligned} \tag{30}$$

we are looking for the solutions of equations (28) in the form

$$\begin{aligned} \vec{E}(t,\vec{x}) &= \vec{E}_A(\vec{x})\cos\omega t + \vec{E}_B(\vec{x})\sin\omega t, \\ \vec{H}(t,\vec{x}) &= \vec{H}_A(\vec{x})\cos\omega t + \vec{H}_B(\vec{x})\sin\omega t. \end{aligned} \tag{31}$$

For the 16 time-independent amplitudes we obtain the following two nonlinked subsystems

$$\begin{aligned} &\operatorname{curl}\vec{H}_A - \omega\varepsilon\vec{E}_B = \operatorname{grad} E^0_A, \\ &\operatorname{curl}\vec{E}_B - \omega\mu\vec{H}_A = -\operatorname{grad} H^0_B, \\ &\operatorname{div}\vec{E}_B = \omega\mu E^0_A, \\ &\operatorname{div}\vec{H}_A = -\omega\varepsilon H^0_B, \end{aligned} \tag{32}$$

$$\begin{aligned} &\operatorname{curl}\vec{H}_B + \omega\varepsilon\vec{E}_A = \operatorname{grad} E^0_B, \\ &\operatorname{curl}\vec{E}_A + \omega\mu\vec{H}_B = -\operatorname{grad} H^0_A, \\ &\operatorname{div}\vec{E}_A = -\omega\mu E^0_B, \\ &\operatorname{div}\vec{H}_B = \omega\varepsilon H^0_A. \end{aligned} \tag{33}$$

Below we consider only the first subsystem (32). It is quite enough because the subsystems (32) and (33) are connected with transformations

$$\begin{aligned} &E \longrightarrow H, \quad H \longrightarrow -E, \\ &\varepsilon E \longrightarrow \mu H, \quad \mu H \longrightarrow -\varepsilon E, \\ &\varepsilon \longrightarrow \mu, \quad \mu \longrightarrow \varepsilon, \end{aligned} \tag{34}$$

which are the generalizations of duality transformation of free electromagnetic field. Due to this fact the solutions of subsystem (33) can be easily obtained from the solutions of subsystem (32).

Furthermore, it is useful to separate equations (32) into the following subsystems:

$$\begin{aligned} &\omega\varepsilon E^3_B - \partial_1 H^2_A + \partial_2 H^1_A + \partial_3 E^0_A = 0, \\ &\omega\varepsilon H^0_B + \partial_1 H^1_A + \partial_2 H^2_A + \partial_3 H^3_A = 0, \\ &-\omega\mu E^0_A + \partial_1 E^1_B + \partial_2 E^2_B + \partial_3 E^3_B = 0, \\ &\omega\mu H^3_A - \partial_1 E^2_B + \partial_2 E^1_B - \partial_3 H^0_B = 0 \end{aligned} \tag{35}$$

and

$$\begin{aligned} &\omega\varepsilon E^1_B - \partial_2 H^3_A + \partial_3 H^2_A + \partial_1 E^0_A = 0, \\ &\omega\varepsilon E^2_B - \partial_3 H^1_A + \partial_1 H^3_A + \partial_2 E^0_A = 0, \\ &\omega\mu H^1_A - \partial_2 E^3_B + \partial_3 E^2_B - \partial_1 H^0_B = 0, \\ &\omega\mu H^2_A - \partial_3 E^1_B + \partial_1 E^3_B - \partial_2 H^0_B = 0. \end{aligned} \tag{36}$$

Assuming the spherical symmetry case, when $\Phi(\vec{x}) = \Phi(r)$, $r \equiv |\vec{x}|$, we are making the transition into the spherical coordinate system and looking for the solutions in the spherical coordinates in the form

$$(E,H)\,(\vec{r}) = R_{(E,H)}\,(r)\,f_{(E,H)}\,(\theta,\phi) \tag{37}$$

where $E \equiv \left(\vec{E}, E^0\right)$, $H \equiv \left(\vec{H}, H^0\right)$. We choose for the subsystem (35) the d'Alembert Ansatz in the form

$$\begin{aligned}
\overline{E}^0_A &= \overline{C}_{E_4} R_{H_4} P_{l_{H_4}} \overline{m}_4 \mathrm{e}^{-\mathrm{i}\overline{m}_4\phi},\\
\overline{E}^k_B &= \overline{C}_{E_k} R_{E_k} P^{\overline{m}_k}_{l_{E_k}} \mathrm{e}^{-\mathrm{i}\overline{m}_k\phi},\\
\overline{H}^0_B &= \overline{C}_{H_4} R_{E_4} P^{\overline{m}_4}_{l_{E_4}} \mathrm{e}^{-\mathrm{i}\overline{m}_4\phi}, \quad k=1,2,3.\\
\overline{H}^k_A &= \overline{C}_{H_k} R_{H_k} P^{\overline{m}_k}_{l_{H_k}} \mathrm{e}^{-\mathrm{i}\overline{m}_k\phi}.
\end{aligned} \tag{38}$$

We use the following representation for ∂_1, ∂_2, ∂_3 operators in spherical coordinates

$$\begin{aligned}
&\partial_1 CRP^m_l \mathrm{e}^{\mp\mathrm{i}m\phi}\\
&= \frac{\mathrm{e}^{\mp\mathrm{i}m\phi}C}{2l+1}\cos\phi\left(R_{,l+1}P^{m+1}_{l-1} - R_{,-l}P^{m+1}_{l+1}\right)\\
&\qquad + \mathrm{e}^{\mp\mathrm{i}(m-1)\phi} C \frac{m}{\sin\theta} P^m_l \frac{R}{r},\\
&\partial_2 CRP^m_l \mathrm{e}^{\mp\mathrm{i}m\phi}\\
&= \frac{\mathrm{e}^{\mp\mathrm{i}m\phi}C}{2l+1}\sin\phi\left(R_{,l+1}P^{m+1}_{l-1} - R_{,-l}P^{m+1}_{l+1}\right)\\
&\qquad \mp \mathrm{e}^{\mp\mathrm{i}(m-1)\phi} C \frac{\mathrm{i}m}{\sin\theta} P^m_l \frac{R}{r},\\
&\partial_3 CRP^m_l \mathrm{e}^{\mp\mathrm{i}m\phi} = \frac{\mathrm{e}^{\mp\mathrm{i}m\phi}C}{2l+1}\left(R_{,l+1}(l+m)P^m_{l-1}\right.\\
&\qquad \left. + R_{,-l}(l-m+1)P^m_{l+1}\right).
\end{aligned} \tag{39}$$

Substitutions (38) and (39) together with the assumptions

$$\begin{aligned}
&R_{E_\alpha} = R_E, && l_{E_\alpha} = l_E,\\
&R_{H_\alpha} = R_H, && l_{H_\alpha} = l_H,\\
&\overline{m}_1 = \overline{m}_2 = \overline{m}_3 - 1 = \overline{m}_4, &&\\
&\overline{C}_{H_1} = \mathrm{i}\overline{C}_{H_2}, && \overline{C}_{E_2} = -\mathrm{i}\overline{C}_{E_1},\\
&\overline{C}_{H_4} = -\mathrm{i}\overline{C}_{E_3}, && \overline{C}_{H_3} = -\mathrm{i}\overline{C}_{E_4},\\
&\overline{C}^I_{H_2} = \overline{C}^I_{E_4}(l^I_H + m + 1), &&\\
&\overline{C}^I_{E_3} = -\overline{C}^I_{E_4} \equiv \overline{C}^I, &&\\
&\overline{C}^I_{H_2} = \overline{C}^I_{E_4}(l^I_H + m + 1) &&\\
&\overline{C}^I_{E_3} = -\overline{C}^I_{E_4} \equiv \overline{C}^I, &&\\
&\overline{C}^I_{E_1} = \overline{C}^I_{E_3}(l^I_E - m), &&\\
&l^I_H = l^I_E - 1 \equiv l^I, &&\\
&\overline{C}^{II}_{H_2} = -\overline{C}^{II}_{E_4}(l^{II}_H - m), &&\\
&\overline{C}^{II}_{E_3} = -\overline{C}^{II}_{E_4} \equiv \overline{C}^{II}, &&\\
&\overline{C}^{II}_{E_1} = -\overline{C}^{II}_{E_3}(l^{II}_E + m + 1), &&\\
&l^{II}_H = l^{II}_E + 1 \equiv l^{II} &&
\end{aligned} \tag{40}$$

into the subsystem (35) guarantee the separation of variables in these equations and lead to the pair of equations for two radial functions R_E, R_H (for the subsystem (36) the procedure is similar):

$$\begin{aligned}
\varepsilon\omega R^I_E - R^I_{H,-l} &= 0,\\
\mu\omega R^I_H + R^I_{E,l+2} &= 0,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\varepsilon\omega R^{II}_E - R^{II}_{H,l+1} = 0,\\
&\mu\omega R^{II}_H + R^{II}_{E,-l+1} = 0;\\
&R_{,a} \equiv \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{a}{r}\right) R.
\end{aligned} \tag{42}$$

For the case $\Phi = -ze^2/r$ (electric and magnetic permeabilities are given in (26)) the equations (41),(42) coincide exactly with the radial equations for the hydrogen atom of the Dirac theory and, therefore, the procedure of their solution is the same as in well-known monographs on relativistic quantum mechanics. It leads to the well-known Sommerfeld-Dirac formula for the fine structure of the hydrogen spectrum. We note only that here the discrete picture of energetic spectrum in the domain $0 < \omega < \mathbf{m}_0 c^2$ is guaranteed by the demand for the solutions of the radial equations (41), (42) to decrease on infinity (when $r \to \infty$). From the equations (41), (42) and this condition the Sommerfeld-Dirac formula

$$\omega = \omega^{hyd}_{nj} = \frac{\mathbf{m}_0 c^2}{\hbar\sqrt{1 + \frac{\alpha^2}{\left(n_r + \sqrt{k^2 - \alpha^2}\right)^2}}} \tag{43}$$

follows, where the notations of the Dirac theory (see, e.g., [43]) are used: $n_r = n - k$, $k = j + 1/2$, $\alpha = e^2/\hbar c$. Let us note once more that the result (43) is obtained here not from the Dirac equation, but from the Maxwell equations (25) with sources (27) in the medium (26).

Substituting (40) into (38) one can easy obtain the angular part of the hydrogen solutions for the $(\vec{E}, \vec{H}, E^0, H^0)$ field and calculate according to (27) the corresponding currents and charges. Let us write down the explicit form for the set of electromagnetic field strengths $(\vec{E}, \vec{H})$, which are the hydrogen solutions of equations (25), and also for the currents and charges generating these field strengths (the complete set of solutions is represented in [1,2]:

$$\begin{aligned}
\vec{E}^I &= R^I_E \begin{vmatrix} (-l+m-1)\, P^m_{l+1}\cos m\phi, \\ (l-m+1)\, P^m_{l+1}\sin m\phi, \\ -P^{m+1}_{l+1}\cos(m+1)\,\phi \end{vmatrix},\\
\vec{H}^I &= R^I_H \begin{vmatrix} (l+m+1)\, P^m_l \sin m\phi, \\ (l+m+1)\, P^m_l \cos m\phi, \\ -P^{m+1}_l \sin(m+1)\,\phi \end{vmatrix},\\
\vec{j}^I_e &= \operatorname{grad} R^I_H P^{m+1}_l \cos(m+1)\,\phi,\\
\vec{j}^I_{mag} &= -\operatorname{grad} R^I_e P^{m+1}_{l+1} \sin(m+1)\,\phi,\\
\rho^I_e &= -\left(\varepsilon R^I_E\right)_{,l+2} P^{m+1}_l \cos(m+1)\,\phi,\\
\rho^I_{mag} &= -\left(\mu R^I_H\right)_{,-l} P^{m+1}_{l+1} \sin(m+1)\,\phi,
\end{aligned} \tag{44}$$

$$
\begin{aligned}
&\vec{E}^{II} = R_E^{II}\left|\begin{array}{l} (l+m)\,P_{l-1}^{m}\cos m\phi \\ (-l-m)\,P_{l-1}^{m}\sin m\phi \\ P_{l-1}^{m+1}\cos(m+1)\phi \end{array}\right|, \\
&\vec{H}^{II} = R_H^{II}\left|\begin{array}{l} (-l+m)\,P_{l}^{m}\sin m\phi \\ (-l+m)\,P_{l}^{m}\cos m\phi \\ -P_{l}^{m+1}\sin(m+1)\,\phi \end{array}\right|, \\
&\vec{j}_e^{\,II} = \mathrm{grad}\, R_H^{II} P_l^{m+1}\cos(m+1)\,\phi, \\
&\vec{j}_{mag}^{\,II} = -\mathrm{grad}\, R_E^{II} P_{l-1}^{m+1}\sin(m+1)\,\phi, \\
&\rho_e^{II} = -\left(\varepsilon R_E^{II}\right)_{,-l+1} P_l^{m+1}\cos(m+1)\,\phi, \\
&\rho_{mag}^{II} = -\left(\mu R_H^{II}\right)_{,l+1} P_{l-1}^{m+1}\sin(m+1)\,\phi.
\end{aligned} \tag{45}
$$

In one of the possible interpretations the states of the hydrogen atom are described by these field strength functions $\vec{E},\vec{H}$ generated by the corresponding currents and charge densities.

It is evident from (25),(27) that currents and charges in (44),(45) are generated by scalar fields (E^0,H^0). Corresponding to (44),(45), (E^0,H^0) solutions of equations (25) are the following:

$$
\begin{aligned}
E^{I0} &= R_H^{I} P_l^{m+1}\cos(m+1)\,\phi, \\
H^{I0} &= R_E^{I} P_{l+1}^{m+1}\sin(m+1)\,\phi, \\
E^{II0} &= R_H^{II} P_l^{m+1}\cos(m+1)\,\phi, \\
H^{II0} &= R_E^{II} P_{l-1}^{m+1}\sin(m+1)\,\phi.
\end{aligned} \tag{46}
$$

As in quantum theory, the numbers $n=0,1,2,...$; $j=k-1/2=l\mp 1/2$ $(k=1,2,...,n)$ and $m=-l,-l+1,...,l$ mark both the terms (43) and the corresponding exponentially decreasing field functions $\vec{E},\vec{H}$ (and E^0,H^0) in (44)–(46), i. e. they mark the different discrete states of the classical electrodynamical field (and the densities of the currents and charges) which by definitions describes the corresponding states of hydrogen atom in the model under consideration.

Note that the radial equations (41),(42) cannot be obtained if one neglects the sources in equations (25), or one (electric or magnetic) of these sources. Moreover, in this case there is no solution effectively concentrated in atomic region.

Now we can show on the basis of this model that the assertions known as *Bohr's postulates are the consequences of equations (25) and of their classical interpretation*, i. e. these assertions can be derived from the model, there is no necessity to postulate them from beyond the framework of classical physics as it was in Bohr's theory. To derive the first Bohr's postulate one can calculate the generalized Pointing vector for the hydrogen solutions (44)–(46), i. e. for the compound system of stationary electromagnetic and scalar fields $(\vec{E},\vec{H},E^0,H^0)$

$$
\vec{P}_{gen} = \int \mathrm{d}^3x(\vec{E}\times\vec{H} - \vec{E}E^0 - \vec{H}H^0). \tag{47}
$$

The straightforward calculations show that not only vector (47) is identically equal to zero but the Pointing vector itself and the term with scalar fields (E^0,H^0) are also identically equal to zero:

$$
\begin{aligned}
\vec{P} &= \int \mathrm{d}^3x(\vec{E}\times\vec{H}) \equiv 0, \\
&\int \mathrm{d}^3x(\vec{E}E^0 + \vec{H}H^0) \equiv 0.
\end{aligned} \tag{48}
$$

This means that in stationary states hydrogen atom does not emit any Pointing radiation neither due to the electromagnetic $(\vec{E},\vec{H})$ field, nor to the scalar (E^0,H^0) field. That is the mathematical proof of the first Bohr postulate.

The similar calculations of the energy for the same system

$$
\mathbf{W} \equiv \frac{1}{2}\int \mathrm{d}^3x(\vec{E}^2 + \vec{H}^2 + E_0^2 + H_0^2) \tag{49}
$$

give a constant $\mathbf{W}_{nl}$, depending on n,l (or n,j) and independent of m. In our model this constant is to be identified with the parameter ω in equations (1) which in the stationary states of $(\vec{E},\vec{H},E^0,H^0)$ field appears to be equal to the Sommerfeld-Dirac value ω_{nj}^{hyd} (43). By abandoning the $\hbar=c=1$ system and putting arbitrary "A" in equations (25) instead of $\hbar$ we obtain final ω_{nj}^{hyd} with "A" instead of $\hbar$. Then the numerical value of $\hbar$ can be obtained by comparison of ω_{nj}^{hyd} containing "A" with the experiment. These facts complete the proof of the second Bohr postulate.

This result means that in this model the Bohr postulates are no longer postulates, but the direct consequences of the classical electrodynamical equation (25). Moreover, this means that together with Dirac or Schrodinger equations we have now the new equation which can be used for finding the solutions of atomic spectroscopy problems. In contradiction to the well-known equations of quantum mechanics our equation is the classical one.

Being aware that few interpretations of quantum mechanics (e.g.: Copenhagen, statistical, Feynman's, Everett's, transactional, see e.g. [44–47]) exist, we are far from thinking that here the interpretation can be the only one (about different models of atom see, e.g., [42]). But the main point is that now the classical interpretation (without probabilities) is possible.

In our papers [1–4] we often tried to develop the classical electrodynamical interpretation of the above considered facts. Now let us emphasize that standard quantum mechanical Dirac's (or spinor classical field-theoretical) interpretation of course is here also possible. In this case the above considered facts only demonstrate in explicit forms the classical electrodynamical aspect of the Dirac equation, which explicit consideration is the main purpose of this paper. The successors of magnetic monopole can try to develop here the monopole interpretation (see [48] for the review and some new ideas about monopole) — we note that there are few interesting possibilities of interpretation but we want to mark first of all the mathematical facts which are more important than different ways of interpretation.

The mathematical grounds of the results under consideration are demonstrated and explained by the next section, see below.

4. The Unitary Relationship Between the Relativistic Quantum Mechanics and Classical Electrodynamics in Medium

Let us consider the connection between the stationary Maxwell equations

$$\begin{aligned} \operatorname{curl}\vec{H} - \omega\varepsilon\vec{E} &= \operatorname{grad} E^0, \\ \operatorname{curl}\vec{E} - \omega\mu\vec{H} &= -\operatorname{grad} H^0, \\ \operatorname{div}\vec{E} &= \omega\mu E^0, \\ \operatorname{div}\vec{H} &= -\omega\varepsilon H^0 \end{aligned} \tag{50}$$

which follow from the system (32) after omitting indices A, B, and the stationary Dirac equation following from the ordinary Dirac equation

$$\left(\mathrm{i}\gamma^\mu\partial_\mu - \mathbf{m}_0 + \gamma^0\Phi\right)\Psi = 0, \quad \Psi \equiv (\Psi^\alpha) \tag{51}$$

with $\mathbf{m}_0 \neq 0$ and the interaction potential $\Phi \neq 0$. Assuming the ordinary time dependence

$$\Psi(x) = \Psi(\vec{x})\mathrm{e}^{-\mathrm{i}\omega t} \Longrightarrow \partial_0\Psi(x) = -\mathrm{i}\omega\Psi(x) \tag{52}$$

for the stationary states and using the standard Pauli-Dirac representation for the γ matrices, one obtains the following system of equations for the components Ψ^α of the spinor Ψ:

$$\begin{aligned} -\mathrm{i}\omega\varepsilon\Psi^1 + (\partial_1 - \mathrm{i}\partial_2)\Psi^4 + \partial_3\Psi^3 &= 0, \\ -\mathrm{i}\omega\varepsilon\Psi^2 + (\partial_1 + \mathrm{i}\partial_2)\Psi^3 - \partial_3\Psi^4 &= 0, \\ -\mathrm{i}\omega\mu\Psi^3 + (\partial_1 - \mathrm{i}\partial_2)\Psi^2 + \partial_3\Psi^1 &= 0, \\ -\mathrm{i}\omega\mu\Psi^4 + (\partial_1 + \mathrm{i}\partial_2)\Psi^1 - \partial_3\Psi^2 &= 0 \end{aligned} \tag{53}$$

where ε and μ are the same as in (26). After substitution in Eqs. (54) instead of Ψ the following column

$$\Psi = \text{column}\left| -H^0 + iE^3, -E^2 + iE^1, E^0 + iH^3, -H^2 + iH^1 \right| \tag{54}$$

one obtains Eqs. (50). A complete set of 8 transformations with the same role was obtained with the help of the Pauli-Gursey symmetry operators [49] in our papers [27, 28].

The relationship (54) may be written down in terms of unitary operator similarly to (8). The further consideration of unitary relationship between the equations (50) and (53) is similar to the procedure of Sec. 1 and may be omitted. The details were published in [1, 2].

The mathematical facts considered here prove the one-to-one correspondence between the solutions of the stationary Dirac equation and the stationary Maxwell equations with 4-currents of gradient-like type. Hence, one can, using (54), write down the hydrogen solutions of the Maxwell equations (25) (or (28)) starting from the well-known hydrogen solutions of the Dirac equation (51), i.e. without special procedure of finding the solutions of the Maxwell equations, see [1, 2].

5. Lamb Shift

It is very useful to consider the case of Lamb shift in the approach presented here. This specific quantum electrodynamical effect (as modern theory asserts) can be described here in the framework of classical electrodynamics of media. In order to obtain Lamb shift one must add to $\Phi(\vec{x}) = -Ze^2/r$ in (26) the quasipotential (known, e. g., from [50], which follows, of course, from quantum electrodynamics)

$$-\frac{Ze^4}{60\pi^2\mathbf{m}_0^2}\delta(r) \tag{55}$$

and solve the equations (25) for such medium similarly to the procedure of Sec. 3 (of course, by another approximation method). Finally one obtains the Lamb shift correction to the Sommerfeld - Dirac formula (43). Such Lamb shift can be interpreted as a pure classical electrodynamical effect. It can be considered here as a consequence of polarization of medium (26) and not of polarization of such abstract concept as vacuum in quantum electrodynamics. Thus, we considered one more interesting classical electrodynamical aspect of the Dirac equation.

6. A Brief Remark About Gravity

The unified theory of electromagnetic and gravitational phenomena may be constructed in the approach under consideration in the following way. The main primary equations again are written as (25) and gravity is considered as a medium in these equations, i. e. the electric ε and magnetic μ permeabilities of the medium are some functions of the gravitational potential Φ_{grav}:

$$\varepsilon = \varepsilon(\Phi_{grav}), \quad \mu = \mu(\Phi_{grav}). \tag{56}$$

Gravity as a medium may generate all the phenomena which in standard Einstein's gravity are generated by Riemann geometry. For example, the refraction of the light beam near a big mass star is a typical medium effect in such a unified model of electromagnetic and gravitational phenomena. The idea of such consideration consists in the following. The gravitational interaction between massive objects may be represented as the interaction with some medium, similarly as here (in Eqs. (25)) the electromagnetic interaction between charged particles is considered.

7. Conclusion

One of the conclusions of our investigation presented here and in [3, 4, 29, 30] is that a field equation itself does not answer the question what kind of particles (Bose or Fermi) is described by this equation. To answer this question one needs to find all the representations of the Poincaré group under which the equation is invariant. If more than one such Poincaré representations are found [3, 4, 29, 30], including the representations with integer and half-integer spins, then the given equation describes both Bose and Fermi particles, and both quantization types (Bose and Fermi) [31,32] of the field function, obeying this equation, satisfy the microcausality condition. The strict group-theoretical ground of this assertion is the following [3, 4]: both slightly generalized Maxwell equations (1) (with $\varepsilon = \mu = 1$) and Dirac equation (7) (with $\mathbf{m}_0 = 0$, $\Phi = 0$) are invariant with respect to three different local representations of Poincaré group, namely the standard spinor, vector and tensor-scalar representations generating by the $(0, 1/2) \otimes (1/2, 0)$, $(1/2, 1/2)$, $(0, 1) \otimes (0, 0)$ representations of the Lorentz SL(2,C) group, respectively.

A few words can be said about the interpretation of the Dirac Ψ function. As follows from the consideration presented here, e.g., from the relationships (8) and (54), the new interpretation of the Dirac Ψ function can be suggested too: Ψ function is the combination of the electromagnetic field strengths $\left(\vec{E}, \vec{H}\right)$ and two scalar fields $\left(E^0, H^0\right)$ generating the electromagnetic sources, i.e. in this case the probability or Copenhagen interpretation of the function Ψ is not necessary. Other interpretations were considered in [39].

Thus, the Dirac equation today is still the interesting object for different investigations. We hope that we show here the relationship between the Dirac and the Maxwell equations and the unknown for many readers classical electrodynamical aspect of the Dirac theory.

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